Contract No. Nonr 839(38)
Project No. NR 061-135

A KINETIC THEORETICAL INVESTIGATION OF A FULLY IONIZED GAS

Part II - Some Aspects of Multiple Collisions

by

Toyoki Koga

January 1965

POLYTECHNIC INSTITUTE OF BROOKLYN

DEPARTMENT
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SUMMARY

The Brownian motion of a test body due to multiple interactions with field particles is investigated within or almost within the framework of Markoff's processes. First, Markoff's processes are studied as presenting such multiple interactions. Based on the study and by means of Markoff's method of random flights, we investigate the Brownian motion of an elastic test body submerged in a rarefied gas constituted of elastic molecules, under the condition that mutual interactions among field particles are negligible. It is shown that there is no difference in effect between temporal repetitions of random binary collisions and multiple collisions (random binary collisions superposed at one moment of time), so far as the friction and diffusion of the test body in momentum space are concerned. The situation is similar when a test body with electric charge is submerged in an electron gas, if the mutual interactions among electrons are ignored.

† This research was supported by the Office of Naval Research under Contract No. Nonr 839(38), Project No. NR 061-135.

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It is not feasible, however, to ignore those mutual interactions of field electrons and to represent electronic multiple interactions by temporal repetitions of random binary interactions, each of which takes place independently: Fluctuations of limitlessly large amplitudes in the spatial distribution of electrons, which may possibly take place in this approximation, do not seem realistic, because a limitless concentration of potential energy accompanying a concentration of electrons in a local spot cannot be permitted. Amplitudes of such fluctuations and/or microscopic disturbances must have a certain maximum limit. [The situation does not change even when the interaction force law is of the Debye-Hückel type.] A kinetic theoretical scheme of treating fully ionized gas in the light of this fact is proposed.

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SECTION I

INTRODUCTION

The difficulty in kinetic theoretical treatments of an electron gas or a fully ionized gas may be attributed mainly to the fact that we are not able to treat precisely more than two body problems. In Part I of this report we showed that methods of treating multiple interactions among electrons in an electron gas in accordance with the B-B-G-K-Y hierarchy are not plausible. In the present part, we attempt, by means of several "mental experiments", to find characteristic effects which distinguish multiple interactions from superposed binary interactions. Let us first suppose that the effect of multiple collisions between a test body and field particles is divided into three parts: (i) the effect of the forces exerted on the test body simultaneously by many field particles, (ii) the effect of reactive forces exerted by the test body on field particles, (iii) the effect of mutual interactions among field particles. Since we do not know, in general, the way to synthesize the total effect from those partial effects, that is, the many-body problem, the classification seems simply conceptual with no definite physical meaning. We note, however, that it is possible to conceive particular conditions under which effects (ii) and (iii) are ignored either completely or partly, and yet interactions are multiple. By studying such particular cases, we may obtain some aspects of statistical effects of the multiplicity of interaction, and, we hope, may find some clue for

considering multiple interactions among charged particles in a fully ionized gas.

We suppose, first, that a test body is an elastic body with linear dimension D which is sufficiently larger than the average distance of two neighboring molecules of the gas in which the test body is submerged. Further, we suppose that D is much shorter than the mean free path of molecules of the gas, and that the mass of the test body denoted by M is much larger than the mass of a molecule m. Under these conditions, the interactions between the test body and field molecules may be multiple. However, the mutual interactions among field particles are negligible. We investigate the statistical behavior of the test body by assuming that the gas is in thermal equilibrium. At first glance, it seems possible to treat the particular problem by means of the usual technics of kinetic theory based on the Liouville equation of those particles. We note, however, that collisions are strong regarding field molecules even though the collisions are weak for the test body. Under this circumstance the coarse-graining of the Liouville equation deriving the statistical behavior of the test body is still complicated. We wish to find some other approach; We see that the collision processes under consideration may be Markoffian or almost Markoffian. Markoff processes have been well studied as a mathematical theory.* The Brownian motion of a test body in a fluid is well-known as a typical example of the theory. Usually, however, the field particles are not objects of direct investigation in the theory: Physical interpretations of the

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*Since Einstein's pioneering study of the Brownian motion[1] in the beginning of this century, the theory has been developed by many authors. The names of Smoluchowski, Fokker, Planck, Ornstein, Burgers, Fürth, Uhlenbeck, and Chandrasekhar are well remembered.[2, 3, 4, 5]

results are achieved by providing data of relevant physical quantities such as friction and diffusion tensors from outside the theory[2, 3]. For instance, the friction coefficient is often provided by Stokes' friction law. One of the primary interests in those studies has been to show that the distribution of the state of a test body finally becomes Maxwellian. It is noted, however, that the theory itself has not proved the assertion that the temperature of the final state of the test body is the same as that of the fluid.*

In this study by taking advantage of the simplicity of collision mode as stated above, we attempt to include the dynamics of those collisions within the framework of the theory of Markoff processes as stated in Section II. In Section III we first consider a special case where the test body and the field molecules are extremely rigid so that the period of each collision is extremely short and hence collisions are binary. It is shown that the temperature of the final distribution of the test body is the same as that of the gas.** We then assume that the period of each collision is finite so that many collisions occur simultaneously. Physical quantities of interest are obtained by means of Markoff's method of random flights under various conditions. In view of the conclusion attained in this section, we consider some aspects of multiple

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*According to the principle of statistical mechanics, we may say so. The question is open, however, if the test body is much larger than molecules.

**Related to the investigation in this section, it is shown that the Boltzmann equation which is obtained under the assumption that collisions are binary and Markoffian, is reduced directly to the Fokker-Planck equation by assuming that interactions are weak. See Appendix B.

interactions in an electron gas in the following section. Finally, a set of basic equations for electron gas is proposed. The detailed analyses of the equations will be given in Part III.

SECTION II

COLLISIONS AS MARKOFF'S PROCESSES

First we remember Markoff's processes, well studied in the mathematical theory. By taking $\vec{p}$ for a variable of state, t for time, and $f_o(\vec{p}, t)$ for the distribution, we assume that the evolution of f_o is a Markoff process. In other words, f_o is assumed to satisfy the Smoluchowski equation:

$$f_o(\vec{p}, t + \Delta t) = \iiint f_o(\vec{p}', t)\ \psi(\vec{p}', t/\vec{p}, t + \Delta t)\ d^3p' \tag{2. 1}$$

where ψ is defined as transition probability. By putting

$$\vec{p}\,' = \vec{p} - \Delta\vec{p} \tag{2. 2}$$

we may write

$$\begin{aligned}\psi(\vec{p}', t/\vec{p}, t + \Delta t)\ d^3p' &= -\ \psi(\vec{p} - \Delta\vec{p}, t/\vec{p} - \Delta\vec{p} + \Delta\vec{p},\ t + \Delta t)\ d^3\Delta p \\ &= \varphi(\vec{p} - \Delta\vec{p}, t; \Delta\vec{p}, \Delta t)\ d^3\Delta p\end{aligned} \tag{2. 3}$$

By expanding φ in a Taylor series, we obtain

$$\begin{aligned}&\psi(\vec{p}\,', t/\vec{p}, t + \Delta t)\ d^3 p' \\ &= \Big[\ \varphi(\vec{p}, t;\Delta\vec{p}, \Delta t) - \Delta\vec{p}\cdot\frac{\partial\varphi}{\partial\vec{p}} + \frac{1}{2!}\Delta\vec{p}\,\Delta\vec{p} : \frac{\partial^2\varphi}{\partial\vec{p}\,\partial\vec{p}} \\ &\quad - \frac{1}{3!}\Delta\vec{p}\,\Delta\vec{p}\,\Delta\vec{p}\ \vdots\ \frac{\partial^3\varphi}{\partial\vec{p}\,\partial\vec{p}\,\partial\vec{p}} + \cdots\Big]\ d^3\Delta p\end{aligned} \tag{2. 4}$$

We put for φ

$$\iiint \varphi(\vec{p},\ t;\ \Delta\vec{p},\ \Delta t)\ d^3\Delta p = 1 \tag{2.5}$$

If we define $<\Delta\vec{p}>_{\Delta t}$, $<\Delta\vec{p}\,\Delta\vec{p}>_{\Delta t}$, ----, by

$$<\Delta\vec{p}>_{\Delta t} = \iiint \Delta\vec{p}\ \varphi(\vec{p}, t;\ \Delta\vec{p},\ \Delta t)\ d^3\Delta p,$$

$$<\Delta\vec{p}\,\Delta\vec{p}>_{\Delta t} = \iiint \Delta\vec{p}\,\Delta\vec{p}\ \varphi\, d^3\Delta p, \tag{2.6}$$

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Eq. (2.1) yields, as shown in Appendix A,

$$\frac{\partial f_o(\vec{p}, t)}{\partial t}\Delta t + \frac{1}{2!}\frac{\partial^2 f_o}{\partial t^2}(\Delta t)^2 + \frac{1}{3!}\frac{\partial^3 f_o}{\partial t^3}(\Delta t)^3 + \cdots$$

$$= -\frac{\partial}{\partial\vec{p}}\cdot(<\Delta\vec{p}>_{\Delta t} f_o) + \frac{1}{2!}\frac{\partial^2}{\partial\vec{p}\,\partial\vec{p}}:(<\Delta\vec{p}\,\Delta\vec{p}>_{\Delta t} f_o)$$

$$-\frac{1}{3!}\frac{\partial^3}{\partial\vec{p}\,\partial\vec{p}\,\partial\vec{p}}\ \vdots\ (<\Delta\vec{p}\,\Delta\vec{p}\,\Delta\vec{p}>_{\Delta t} f_o) + \cdots \tag{2.7}$$

It is noted that $<\Delta\vec{p}>_{\Delta t}$, $<\Delta\vec{p}\Delta\vec{p}>_{\Delta t}$, ---- are, in general, functions of Δt. If $<\Delta p>_{\Delta t}/\Delta t$, $<\Delta p\ \Delta p>_{\Delta t}/\Delta t, \ldots$ have certain limit values, as Δt tends to decrease to a short period denoted by τ, we may write for (2.7)

$$\frac{\partial f_o}{\partial t} = -\frac{\partial}{\partial\vec{p}}\cdot(<\Delta\vec{p}> f_o) + \frac{1}{2!}\frac{\partial^2}{\partial\vec{p}\,\partial\vec{p}}:(<\Delta\vec{p}\,\Delta\vec{p}> f_o)$$

$$-\frac{1}{3!}\frac{\partial^3}{\partial\vec{p}\,\partial\vec{p}\,\partial\vec{p}}\ \vdots\ (<\Delta\vec{p}\,\Delta\vec{p}\,\Delta\vec{p}> f_o) + \cdots \tag{2.8}$$

where

$$<\Delta\vec{p}> = \lim_{\Delta t \to \tau} \frac{<\Delta\vec{p}>_{\Delta t}}{\Delta t}, \qquad \text{(independent of } \tau)$$

$$<\Delta\vec{p}\,\Delta\vec{p}> = \lim_{\Delta t \to \tau} \frac{<\Delta\vec{p}\,\Delta\vec{p}>_{\Delta t}}{\Delta t}, \quad \text{(independent of } \tau) \tag{2.9}$$

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If they fail to have certain limit values, it is difficult to give proper physical interpretations of Eq. (2.7).

Secondly, we consider the same process as a physical problem. To this end, we consider a test body whose momentum changes as a function of time due to forces exerted at random from the outside. By considering similar systems, we may define the distribution function of the test body f_o, as a function of its momentum and time. Each of the external foces, denoted by $\vec{\mathcal{F}}_r$, is assumed to appear with its proper probability w_r with respect to a unit time, and to continue for its proper period τ_r. The change of the momentum of the test body due to $\vec{\mathcal{F}}_r$ is given by

$$\vec{p}_r = \int_{t_r}^{t_r + \tau_r} \vec{\mathcal{F}}_r \, dt \tag{2.10}$$

where t_r is the initial time when $\vec{\mathcal{F}}_r$ begins to appear. We assume that w_r is uniform during an elementary period Δt which is longer than τ_r

$$\Delta t > \tau_r \tag{2.11}$$

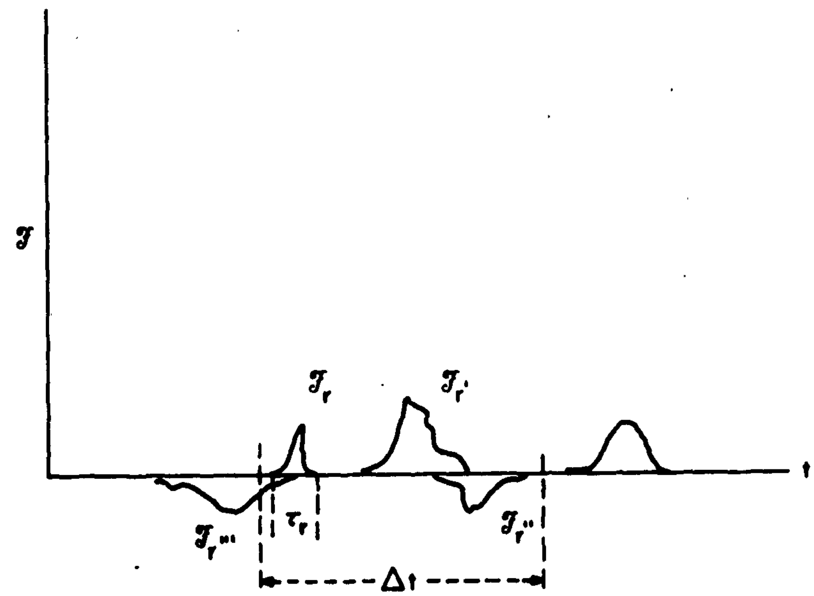

Fig. 1. Force $\vec{\mathcal{T}}_r$ begins at $t = t_r$ and ends at $t = t_r + \tau_r$. $\vec{\mathcal{T}}_{r'''}$ is partly outside the period Δt. Therefore the process is not completely Markoffian.

We also assume

$$w_r \tau_r << 1 . \qquad (2.12)$$

In this section, we do not consider the physical cause of each force; we simply assume that $\vec{\mathcal{F}}_r$ and w_r do not change due to the change of momentum of the test body during Δt. To apply the result of the present section to physical problems, it is necessary to choose cases where the assumption is satisfied.

The number of forces which occur during the period Δt is given by

$$<\nu>_{\Delta t} = \sum_{r=1}^{N} w_r \Delta t \qquad (2.13)$$

where N is the total number of forces and it is assumed that a force does not appear twice, due to some mechanism of the external cause. It is possible that a force exists partly within the period Δt and partly beyond (before or after) the period. Due to (2.11) and(2.12), however, the number of such forces is negligibly small, otherwise the processes due to these forces are not Markoffian. Under those conditions, the probable value of $\Delta\vec{p}_{\Delta t}$, the momentum gain of the test body in Δt, is

$$<\Delta\vec{p}>_{\Delta t} = \sum_{r=1}^{N} \vec{p}_r \, w_r \, \Delta t \qquad (2.14)$$

It is noted that $\Delta\vec{p}_{\Delta t}$ fluctuates, if similar experiments are repeated. The probability distribution of $\Delta\vec{p}_{\Delta t}$ is denoted by

$$\varphi(\Delta\vec{p}; \vec{p}) \qquad (2.15)$$

where $\vec{p}$ is the momentum of the test body.

Of course we have

$$\iiint \varphi(\Delta\vec{p};\ \vec{p})\ d^3\Delta p = 1 \tag{2.16}$$

According to the definitions of f_o and φ, we write

$$f_o(\vec{p}, t + \Delta t) = \iiint \varphi(\Delta\vec{p};\ \vec{p}-\Delta\vec{p})\ f_o(\vec{p}-\Delta\vec{p}, t)\ d^3\Delta p \tag{2.17}$$

This is obviously equivalent to Eq. (2. 1), and hence Eq. (2. 17) yields Eq. (2. 7).

In the following, we consider $\varphi(\Delta\vec{p};\vec{p})$ itself. The probability of no force appearing in Δt is

$$S(0) = \prod_{r=1}^{N} (1 - w_r \Delta t). \tag{2.18}$$

The probability of only one force appearing in Δt is

$$S(1) = \sum_{r=1}^{N} s_r = \sum_{r=1}^{N} w_r \Delta t \prod_{r' \neq r} (1 - w_{r'} \Delta t)$$

Similarly,

$$S(2) = \sum_{r_1 < r_2}\sum s_{r_1 r_2} = \sum_{r_1 < r_2}\sum w_{r_1} \Delta t\, w_{r_2} \Delta t \prod_{r_3 \neq r_1, r_2} (1 - w_{r_3} \Delta t)$$

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$$\begin{aligned} S(n) &= \sum\sum_{r_1 < r_2 < \cdots < r_n} \cdots \sum s_{r_1 r_2 \cdots r_n} \\ &= \sum\sum_{r_1 < r_2 \cdots < r_n} \cdots \sum w_{r_1} w_{r_2} \cdots w_{r_n} (\Delta t)^n \prod_{r_{n+1} \neq r_1, r_2, \cdots, r_n} (1 - w_{r_{n+1}} \Delta t) \end{aligned} \tag{2.19}$$

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Case I. We first consider the case where

$$\langle \nu \rangle_{\Delta t} = \sum_{N >> 1} w_r \, \Delta t < 1, \tag{2.20}$$

In this case, we may expect that

$$s_r = w_r \, \Delta t << 1 \tag{2.21}$$

and the higher powers of $w_r \, \Delta t$ are negligible. Hence S(0) given by (2. 18) and S(1) predominate. We may put for $\langle \Delta\vec{p} \, \Delta\vec{p} \rangle \Delta t$

$$\langle \Delta\vec{p} \Delta\vec{p} \rangle_{\Delta t} = \sum s_r \, \vec{p}_r \, \vec{p}_r \tag{2.22}$$

Similarly,

$$\langle \Delta\vec{p} \, \Delta\vec{p} \, \Delta\vec{p} \rangle_{\Delta t} = \sum s_r \, \vec{p}_r \, \vec{p}_r \, \vec{p}_r \tag{2.23}$$

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In order to satisfy condition (2. 21) under the restriction (2. 11), it is necessary to assume that either τ_r is extremely short or w_r is very small. When a rigid test body is submerged in a rarefied gas composed of rigid molecules, τ_r may be extremely short and w_r may be small. *

*The Boltzmann equation is derived under the same conditions. Hence it is expected that the Boltzmann equation may be reduced to (2. 8). This is shown in Appendix (B).

<u>Case II.</u> If we assume

$$<\nu>_{\Delta t} = \sum w_r \,\Delta t \;>> 1 \tag{2.24}$$

we have to consider all the S's given by (2.18) and (2.19). In this case, $<\Delta\vec{p}>_{\Delta t}$ given by (2.14) is still valid: we have, however, for $<\Delta\vec{p}\,\Delta\vec{p}>_{\Delta t}$

$$<\Delta\vec{p}\,\Delta\vec{p}>_{\Delta t} = \sum_{r=1}^{N} s_r\,\vec{p}_r\,\vec{p}_r + \sum\sum_{r_1<r_2} s_{r_1 r_2}(\vec{p}_{r_1}+\vec{p}_{r_2})(\vec{p}_{r_1}+\vec{p}_{r_2})$$

$$+ \cdots + \sum\sum\cdots\sum_{r_1<r_2<\cdots<r_n} s_{r_1 r_2 \cdots r_n}(\vec{p}_{r_1}+\cdots+\vec{p}_{r_n})(\vec{p}_{r_1}+\cdots+\vec{p}_{r_n})$$

$$+ \cdots\cdots. \tag{2.25}$$

Considering the law of large number, it is expected that, as $<\nu>_{\Delta t}$ increases, the fluctuation in the number $\nu_{\Delta t}$ decreases* and we may simply take

$$<\Delta\vec{p}\,\Delta\vec{p}>_{\Delta t} = \sum\sum\cdots\sum_{r_1<r_2<\cdots<r_{\bar{n}}} s_{r_1 \cdots r_{\bar{n}}}(\vec{p}_{r_1}+\cdots+\vec{p}_{r_{\bar{n}}})(\vec{p}_{r_1}+\cdots+\vec{p}_{r_{\bar{n}}})$$

$$\bar{n} = <\nu>_{\Delta t} \tag{2.26}$$

We may consider $<\Delta\vec{p}\Delta\vec{p}\,\Delta\vec{p}>_{\Delta t_1}$ ---- in a similar way. Those quantities so far obtained do not necessarily have definite physical meanings: For those quanti-

*By taking Δt extremely long, we may consider the forces appearing in Δt as a "population". In the present argument, we are doing a random sampling from the population. See any text book of "Statistics".

ties to have certain physical meanings, it is necessary that

$$\langle \nu \rangle = \frac{\langle \nu \rangle_{\Delta t}}{\Delta t}, \quad \langle \Delta\vec{p} \rangle = \frac{\langle \Delta\vec{p} \rangle_{\Delta t}}{\Delta t}, \quad \langle \Delta\vec{p}\,\Delta\vec{p}\,\Delta\vec{p} \rangle = \frac{\langle \Delta p\,\Delta p\,\Delta p \rangle_{\Delta t}}{\Delta t},$$

$$\langle \Delta\vec{p}\,\Delta\vec{p}\,\Delta\vec{p}\,\Delta\vec{p} \rangle = \frac{\langle \Delta\vec{p}\,\Delta\vec{p}\,\Delta\vec{p}\,\Delta\vec{p} \rangle_{\Delta t,}}{\Delta t} \text{ -------} \tag{2.27}$$

must converge to certain values which are independent of Δt, as Δt decreases toward its lower limit given by (2.11). It is also necessary that

$$\frac{\partial^2 f_o}{\partial t^2}\Delta t \Big/ \frac{\partial f_o}{\partial t} << 1, \text{ --------.} \tag{2.28}$$

at this limit.

We simply assumed that w_r and $\vec{\mathcal{J}}_r$ are invariant regardless of the sequence of appearance of more than one force in Δt. As a matter of physics, this assumption is feasible only under certain conditions. In the following sections, we will pay special attention to this assumption in each case so that the assumption will be feasible.

SECTION III

ELASTIC TEST BODIES IN A RAREFIED GAS

Our first question is as follows: Is there any difference between the effect of binary interaction and that of multiple interaction when mutual interactions among field particles are ignorable? In order to answer this question, we investigate several physically conceivable examples within the framework of the restrictive conditions considered in the preceding section. We consider a mechanically elastic test body of linear dimension D and with mass M, submerged in a gas composed of spherical particles (molecules) of one species with diameter σ and mass m, in thermal equilibrium. We assume

$$M >> m, \tag{3.1}$$

$$D >> n^{-\frac{1}{3}} >> \sigma, \tag{3.2}$$

$$\lambda >> D \tag{3.3}$$

where λ is the mean free path of field particles (molecules), n the number density of field particles. It is noted that $n^{-\frac{1}{3}}$ is of the same order as that of the average distance between two neighboring particles. By (3.1) it is assured that the collision between the test body and a field particle is <u>weak</u> so far as the test body is concerned: by (3.2) the number of collisions $\nu_{\Delta t}$ in Δt may be large, and by (3.3) the effect of mutual interactions among field particles are negligible. Due to the last condition, field particles are molecular-disordered.

1. Binary Collision ($\tau_r \to 0$)

Case 1-a. For the convenience of comparison, we first consider binary collision. This condition may be realized by assuming that the test body and field particles are extremely rigid so that the period of a collision is extremely short or $\tau_r = 0$. By taking a sphere of radius D for the test body, as obtained in Appendix C, the average number of field particles colliding with the test body in Δt, which satisfies (2.21) $\sum w_r \, \Delta t < 1$, is given by

$$<\nu>_{\Delta t} = 2\pi^{1/2} n \left(\frac{2kT}{m}\right)^{1/2} R^2 \Delta t \tag{3.4}$$

where

$$R = \frac{D}{2} + \frac{\sigma}{2}, \tag{3.5}$$

n = number density of molecules.

The average momentum given by those particles to the test body is

$$<\Delta\vec{p}>_{\Delta t} = -\frac{4}{3} m \vec{v} <\nu>_{\Delta t} \tag{3.6}$$

where $\vec{v}$ is the velocity of the test body relative to the gas. Further we obtain, according to (2.22),

$$<\Delta\vec{p}\,\Delta\vec{p}>_{\Delta t} = \frac{4}{3} m^2 \left(\frac{2kT}{m}\right) <\nu>_{\Delta t} \vec{\delta}, \tag{3.7}$$

$\vec{\delta}$: unit tensor

In those derivations, it is assumed that

$$\frac{mv^2}{kT} << 1 . \tag{3.8}$$

This assumption may be justified by considering (3.1). Similarly, according to (2.23), we have

$$\left|\text{Components of } \langle \Delta\vec{p}\,\Delta\vec{p}\,\Delta\vec{p} \rangle_{\Delta t}\right| \propto m^3 \left(\frac{2kT}{m}\right) \vec{v}\,\Delta t,$$

$$\left|\text{Components of } \langle \Delta\vec{p}\,\Delta\vec{p}\,\Delta\vec{p}\,\Delta\vec{p} \rangle_{\Delta t}\right| \propto m^4 \left(\frac{2kT}{m}\right)^2 \Delta t \qquad (3.9)$$

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By considering that

$$\frac{\partial f_o}{\partial \vec{p}} / f_o \approx 0 \left[\frac{v}{kT}\right]$$

$$\frac{\partial^2 f_o}{\partial \vec{p}\,\partial \vec{p}} / f_o \approx 0 \left[\frac{1}{MkT} + \left(\frac{v}{kT}\right)^2\right]$$

$$\frac{\partial^3 f_o}{\partial \vec{p}\,\partial \vec{p}\,\partial \vec{p}} / f_o \approx 0 \left[\left(\frac{1}{MkT} + \frac{v^2}{(kT)^2}\right)\left(\frac{v}{kT}\right)\right]$$

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we may ignore those higher order moments as given by (3.9). According to these considerations, Eq. (2.7) yields

$$\frac{\partial f_o}{\partial t} = -\frac{\partial}{\partial \vec{p}} \cdot \left(-\frac{4}{3} m \vec{v} \frac{\langle \nu \rangle_{\Delta t}}{\Delta t}\right) f_o$$

$$+ \frac{1}{2} \frac{\partial^2}{\partial \vec{p}\,\partial \vec{p}} : \left(\frac{4}{3} m^2 \frac{2kT}{m} \frac{\langle \nu \rangle_{\Delta t}}{\Delta t} \vec{\delta}\right) f_o \qquad (3.10)$$

It is easily shown that the right-hand side of the equation vanishes if we put

$$f_o = \alpha_o \left(\frac{M}{2\pi kT}\right)^{3/2} \exp\left[-\frac{M}{2kT} \frac{p^2}{M^2}\right], \qquad (3.11)$$

$$\vec{p} = M\vec{v}.$$

Case $1\text{-}b_1$. For the convenience of later comparison, we here take for the test body a rigid cube moving with velocity $\vec{v}$ which is perpendicular to one of the faces of the cube. In this case, D is the length of a side. The results obtained in Appendix C by taking $\vec{v} = (v, 0, 0)$ are

$$<\nu>_{\Delta t} = \frac{3}{\sqrt{\pi}} \left(\frac{2kT}{m} \right)^{1/2} n\, D^2 \Delta t, \tag{3.12}$$

$$<\Delta p_x>_{\Delta t} = -\frac{4}{3} m\, v <\nu>_{\Delta t},$$

$$<\Delta p_y>_{\Delta t} = <\Delta p_z>_{\Delta t} = 0 \tag{3.13}$$

$$<\Delta p_x \Delta p_x>_{\Delta t} = <\Delta p_y \Delta p_y>_{\Delta t} = <\Delta p_z \Delta p_z>_{\Delta t}$$

$$= \frac{4}{3} m^2 \left(\frac{2kT}{m} \right) <\nu>_{\Delta t} . \tag{3.14}$$

Since Δp_x, Δp_y, and Δp_z are events independent of each other*, it is natural that

$$<\Delta p_x \Delta p_y>_{\Delta t} = <\Delta p_x>_{\Delta t} <\Delta p_y>_{\Delta t} = 0 \tag{3.15}$$

$$<\Delta p_x \Delta p_z>_{\Delta t} = 0$$

Taking advantage of the simplicity of treatments, we obtain

$$<\Delta p_x \Delta p_x \Delta p_x>_{\Delta t} = \frac{-32}{3} m^3 \left(\frac{2kT}{m} \right) v <\nu>_{\Delta t}$$

$$<\Delta p_x \Delta p_x \Delta p_x \Delta p_x>_{\Delta t} = \frac{32}{3} m^4 \left(\frac{2kT}{m} \right)^2 <\nu>_{\Delta t} \tag{3.16}$$

*The particles which contribute to Δp_x are different from the particles which contribute to Δp_y .

The other components of $<\Delta\vec{p}\,\Delta\vec{p}\,\Delta\vec{p}>$ and $<\Delta\vec{p}\,\Delta\vec{p}\,\Delta\vec{p}\,\Delta\vec{p}>$ are shown to vanish. By comparing (3.12), (3.13), and (3.14) respectively with (3.4), (3.6), and (3.7), we may say that the above two cases are similar to each other. In other words, Eq. (3.10) is valid in this case too.

Case 1-b_2. On each of the faces of the cube which are parallel to $\vec{v}$, we assign $<\nu>_{\Delta t}/6$ field particles (with no fluctuation) as colliding on the face in Δt

On the face which is toward the direction of $\vec{v}$, $\left(\frac{<\nu>\Delta t}{6} + \frac{v}{2} D^2 n \Delta t\right)$ field particles are assigned*, and on the opposite face, $\left(\frac{<\nu>\Delta t}{6} - \frac{v}{2} D^2 n \Delta t\right)$ particles are assigned. The distribution in the configuration space is then no longer at random. Assuming that those particles are distributed according to the Maxwell function in the momentum space, we obtain, at the limit $\tau_r = \Delta t = 0$,

$$\left.\begin{aligned} &<\Delta\vec{p}>_{\Delta t} = -\frac{4}{3} m\vec{v} <\nu>_{\Delta t}, \\ &<\Delta p_x \, \Delta p_x>_{\Delta t} = \frac{8}{3} \; mkT <\nu>_{\Delta t}, \\ &<\Delta p_y \, \Delta p_y>_{\Delta t} = \cdots = 0. \end{aligned}\right\} \tag{3.17}$$

Case 1-b_3. We now assume that the distribution of field particles in the configuration space is at random. However, each particle is assumed to have the same magnitudes of momentum components

*See Appendix C, These are the average numbers of particles colliding on the two faces which are perpendicular to $\vec{v}$ in Case 1-b_1.

$$|p_x| = |p_y| = |p_z| = \frac{\iiint m |c_x| f d^3 c}{\iiint f d^3 c}, \tag{3.18}$$

$$= \left(\frac{m k T}{2\pi}\right)^{1/2}$$

In this case we obtain for $\vec{v} = (v, o, o)$

$$\langle \nu \rangle_{\Delta t} = \frac{6nD^2}{(2\pi)^{1/2}} \left(\frac{kT}{m}\right)^{1/2} \Delta t,$$

$$\langle \Delta p_x \rangle_{\Delta t} = - \frac{4}{3} m v \langle \nu \rangle_{\Delta t}, \tag{3.19}$$

$$\langle \Delta p_y \rangle_{\Delta t} = \langle \Delta p_z \rangle_{\Delta t} = 0$$

$$\langle \Delta p_x \Delta p_x \rangle = \langle \Delta p_y \Delta p_y \rangle = \langle \Delta p_z \Delta p_z \rangle$$

$$= (2|p_x|)^2 \langle \nu \rangle_{\Delta t} / 3 = \frac{2}{3\pi} mkT \langle \nu \rangle_{\Delta t}.$$

Case 1-b_4. Suppose that by some mechanism controlling the field particles, the number of field particles colliding on each face of the cube is assigned as in Case 1-b_2, and further each particle has the same momentum components as given by (3.18). In this case, $\langle \nu \rangle_{\Delta t}$ and $\langle \Delta \vec{p} \rangle_{\Delta t}$ are respectively the same as those given in Case 1-b_3. However, $\Delta \vec{p}$ shows no fluctuation, and $\langle \Delta \vec{p} \Delta \vec{p} \rangle_{\Delta t}$ completely vanishes.

2. Multiple Collisions ($\tau_r \neq 0$)

The conditions are similar to those considered in Cases 1, except that the test body and the field particles are not extremely rigid and τ_r, the period of a collision, is finite. In this case the choice of Δt is limited by (2.11) and $<\nu>_{\Delta t}$ given by (2.13) may be much larger than unity:

$$<\nu>_{\Delta t} = \sum w_r \Delta t >> 1 \tag{2.24}$$

In this case, Eq. (2.14) is still valid

$$<\Delta \vec{p}>_{\Delta t} = \sum \vec{p}_r \; w_r \Delta t . \tag{2.14}$$

This is the same as what was obtained in Case 1. For $<\Delta \vec{p} \Delta \vec{p}>_{\Delta t}$, however, we must take (2.26) instead of (2.22).

$$<\Delta \vec{p} \Delta \vec{p}>_{\Delta t} = \sum \cdots \sum \mathscr{J} r_1 \cdots r_n \, (\vec{p}_{r_1} + \cdots + \vec{p}_{r_n}) (\vec{p}_{r_1} + \cdots + \vec{p}_{r_n}) \tag{2.26}$$

Case 2-a. First we consider a spherical body with diameter D for the test body. As in Case 1-a, we have

$$<\nu>_{\Delta t} = 2\pi^{1/2} n \left(\frac{2\,kT}{m} \right)^{1/2} R^2 \Delta t,$$

$$<\Delta \vec{p}>_{\Delta t} = - \frac{4}{3} m \vec{v} <\nu>_{\Delta t} \tag{3.20}$$

For calculating $<\Delta \vec{p} \Delta \vec{p}>_{\Delta t}$, $<\Delta \vec{p} \Delta \vec{p} \Delta \vec{p}>_{\Delta t}, \ldots$, it is convenient to use Markoff's method of free flights as briefly explained in Appendix D. By this method, as

shown in Appendix E, the probability of $\Delta\vec{p}$ being between $\vec{P}$ and $\vec{P} + d\vec{P}$ is given by

$$W(\vec{P})d^3P = \left\{ \frac{1}{8\pi^{3/2}(\frac{4}{3}<\nu>_{\Delta t}\, mkT)^{3/2}} + 0\left[\left(\frac{1}{<\nu>_{\Delta t}}\right)^{7/2}\right]\right\}$$

$$\times \exp\left\{ - \frac{P^2}{\frac{16}{3}<\nu>_{\Delta t}\, mkT} \right\} d^3P \qquad (3.21)$$

See Appendix E, Eq. (E. 9). By means of $W(\vec{P})$, we obtain

$$<\Delta p_x\, \Delta p_x>_{\Delta t} = <\Delta p_y\, \Delta p_y>_{\Delta t} = <\Delta p_z\, \Delta p_z>_{\Delta t}$$

$$= \frac{8}{3}\, mkT <\nu>_{\Delta t} . \qquad (3.22)$$

as shown in Appendix E. So far as $<\nu>_{\Delta t}$, $<\Delta\vec{p}>_{\Delta t}$ and $<\Delta\vec{p}\,\Delta\vec{p}>_{\Delta t}$ are concerned, there is no difference between two cases, one where $<\nu>_{\Delta t} < 1$ and the other $<\nu>_{\Delta t} >> 1$. However, as obtained in Appendix E, in the latter $<\Delta\vec{p}\Delta\vec{p}\,\Delta\vec{p}\,\Delta\vec{p}>$ is proportional to $<\nu>^2_{\Delta t}$. It is not simple to calculate $<\Delta\vec{p}\,\Delta\vec{p}\,\Delta\vec{p}>$, since this involves $\vec{v}$, and hence $A(\vec{\rho})$ is not uniform with respect to the direction of $\vec{\rho}$. In the following cases,where a cube is taken for the test body, we calculate those quantities more easily.

<u>Case 2-b$_1$</u>. In order to make treatments easier, we consider the same test body as in Case 1-b; that is, a cube. By so doing, as shown in Appendix E, we obtain, by assuming $\vec{v} = (v, 0, 0)$,

$$<\nu>_{\Delta t} = \frac{3}{\sqrt{\pi}} D^2 n \left(\frac{2kT}{m}\right)^{1/2} \Delta t,$$

$$<\Delta\vec{p}>_{\Delta t} = -\frac{4}{3} m \vec{v} <\nu>_{\Delta t},$$

$$<\Delta p_x \Delta p_x>_{\Delta t} = <\Delta p_y \Delta p_y>_{\Delta t} = <\Delta p_z \Delta p_z>_{\Delta t} = \frac{8}{3} mkT <\nu>_{\Delta t}, \qquad (3.23)$$

$$<\Delta p_x \Delta p_y>_{\Delta t} = <\Delta p_y \Delta p_z>_{\Delta t} = - - - - - = 0$$

and again

$$<\Delta p_x \Delta p_x \Delta p_x \Delta p_x>_{\Delta t} \propto (<\nu>_{\Delta t})^2 \qquad (3.24)$$

So far as $<\Delta\vec{p}>_{\Delta t}$ and $<\Delta\vec{p}\,\Delta\vec{p}>_{\Delta t}$ are concerned, there is no difference between the effect of binary collision and the effect of multiple collision. At first glance, one might be puzzled. We note, however, that in these two preceding cases, 2-a, 2-b_1, the distribution of field particles colliding with the test body is at random twofold: (i) the random distribution in the configuration space (on the surface of the test body); (ii) the random distribution in the momentum space. It is possible to conceive that these disorders, or one of them, may cause such results. In order to investigate the situation, we calculate $<\Delta\vec{p}\,\Delta\vec{p}>$ in the following three cases.

<u>Case 2-b_2</u>. On each of the faces of the cube which are parallel to $\vec{v}$, $\frac{<\nu>_{\Delta t}}{6}$ field particles (with no fluctuation) are assigned as colliding in Δt; on the face which is faced in the direction of $\vec{v}$, $\left(\frac{<\nu>_{\Delta t}}{6} + \frac{v}{2} D^2 n \Delta t\right)$ field particles*,

*These numbers are the same as in Case 2-b_1. See Appendix E.

are assigned, and on the opposite face, $\left(\frac{<\nu>_{\Delta t}}{6} - \frac{v}{2} D^2 n \Delta t\right)$ field particles are assigned. The distribution in the configuration space is then no longer at random. Assuming that those particles are distributed according to the Maxwell function in the momentum space, we obtain again

$$<\Delta\vec{p}>_{\Delta t} = -\frac{4}{3} m \vec{v} <\nu>_{\Delta t}$$

$$<\Delta p_x \Delta p_x>_{\Delta t} = <\Delta p_y \Delta p_y>_{\Delta t} = <\Delta p_z \Delta p_z>_{\Delta t} = \frac{8}{3} mkT <\nu>_{\Delta t} \tag{3.25}$$

$$<\Delta p_x \Delta p_y>_{\Delta t} = - - - - - - = 0$$

These are exactly the same as given by (3.24).

Case 2-b$_3$. We now assume that the distribution of field particles in the configuration space is at random. However, each particle is assumed to have the same magnitudes of momentum components

$$|p_x| = |p_y| = |p_z| = \iiint m \, |c_x| \, f d^3c \Big/ \iiint f d^3c = (mkT/2\pi)^{1/2} \tag{3.26}$$

In this case we obtain for $\vec{v} = (v, 0, 0)$

$$<\nu>_{\Delta t} = \frac{6\, nD^2}{(2\pi)^{1/2}} \left(\frac{kT}{m}\right)^{1/2} \Delta t \tag{3.27}$$

$$<\Delta\vec{p}>_{\Delta t} = -\frac{4}{3} m \vec{v} <\nu>_{\Delta t} \tag{3.28}$$

$$\langle \Delta p_x \Delta p_x \rangle_{\Delta t} = \langle \Delta p_y \Delta p_y \rangle_{\Delta t} = \langle \Delta p_z \Delta p_z \rangle_{\Delta t}$$

$$= \frac{2}{3\pi} mkT \langle \nu \rangle_{\Delta t} \tag{3.29}$$

$$\langle \Delta p_x \Delta p_y \rangle_{\Delta t} = - - - - - - = 0$$

Considering the relation between $\langle \Delta\vec{p} \rangle_{\Delta t}$ and $\langle \Delta\vec{p}\, \Delta\vec{p} \rangle_{\Delta t}$ we note that the temperature of the test body in equilibrium is no longer the same as that of the gas. This conclusion is conceivable since we ignored the Maxwell distribution of particles colliding with the test body.

Case 2-b_4. Finally we assume that the number of particles colliding on each face of the cube is assigned as in Case 2-b_2, and further each particle has the same momentum component as given by (3.26). In this case $\langle \nu \rangle_{\Delta t}$ and $\langle \Delta\vec{p} \rangle_{\Delta t}$ are respectively the same as those given in Case 2-b_3. However, $\Delta\vec{p}$ has no fluctuation, and $\langle \Delta\vec{p}\, \Delta\vec{p} \rangle_{\Delta t}$ completely vanishes.

These results, obtained so far through rather primitive (but precise) treatments, reveal some aspects of multiple interaction. So far as $\langle \nu \rangle_{\Delta t}$, $\langle \Delta\vec{p} \rangle_{\Delta t}$ and $\langle \Delta\vec{p}\, \Delta\vec{p} \rangle_{\Delta t}$ are concerned, there is no difference in effect between binary collision and multiple collision. The binary interaction is based on the hypothesis of molecular disorder. Our treatment of multiple interaction is also carried out by granting the same assumption. The assumption is justified by assuming that mutual interactions among field particles are negligible. As a matter of physics, the binary collision assumption and the assumption of no mutual interaction among field particles are compatible. It is noted, however,

that the multiple collision assumption and the assumption of no mutual interaction among field particles are not necessarily compatible, as considered in the following section.

Another important aspect is that $<\Delta\vec{p}>_{\Delta t}$, (and also $<\Delta\vec{p}\Delta\vec{p}\Delta\vec{p}>_{\Delta t}$), is directly related to the non-uniform distribution of field particles reflected from the surface of the test body: In other words, $<\Delta\vec{p}>_{\Delta t}$ is said to be caused by the wake produced behind the test body. On the other hand, the wake has no effect on $<\Delta\vec{p}\Delta\vec{p}>_{\Delta t}$, so far as $mv^2/kT << 1$: $<\Delta\vec{p}\Delta\vec{p}>_{\Delta t}$ is caused simply by fluctuations which appear in $\Delta\vec{p}$, and are smoothed out in $<\Delta\vec{p}>_{\Delta t}$. See the details of calculation in Appendix E:

SECTION IV

AN ELECTRICALLY CHARGED TEST BODY IN ELECTRON GAS

In Section III, we obtained experimentally a rule that multiple interactions are equivalent to superposed binary interactions, when mutual interactions among field particles are ignored and so far as the friction and the diffusion (in the momentum space) of the test body are concerned. In this section, we assume that the rule is valid for considering interactions among charged particles.

We consider for the test body a particle with electric charge Q and mass M. The test body is submerged in an electron gas; a positive charge is

assumed to be continuously and uniformly spread in the space so that the positive charge neutralizes the charge of the whole electrons. We also assume

$$M \gg \text{electronic mass } m \qquad \text{(i)}$$

$$Q \lesssim \text{ electronic charge } e \qquad \text{(ii)}$$

and

$$mv^2/kT \ll 1 \qquad \text{(iii)}$$

where T is the temperature of the electron gas which is in thermal equilibrium, $\vec{v}$ the speed of the test body relative to the electron gas.

As stated in the Introduction, the interaction among those particles is due to (a) the forces exerted on the test body by field particles (electrons), (b) the forces exerted on field particles by the test body, (c) forces among field particles

Approximation A

Let us first ignore (b) and (c) and see the consequence of applying the rule set forth at the beginning of this section.

Effect 1. $\langle \Delta\vec{p} \rangle_{\Delta t} = 0$. Consider the effect of an electron, with velocity $\vec{c}$ which is parallel to $\vec{v}$, encountering the test body with impact vector $\vec{r}$. [Here impact vector is defined as impact parameter together with its direction.] We may consider another electron with the same velocity with impact vector $-\vec{r}$. The effects cancel each other. Next we consider an electron, with velocity $\vec{c}$ which is perpendicular to $\vec{v}$, passing by the test body with impact

parameter $\vec{r}$. We may consider another electron with the same velocity and with impact parameter $-\vec{r}$. The effects of these two electrons cancel each other. See Fig. 2. Considering

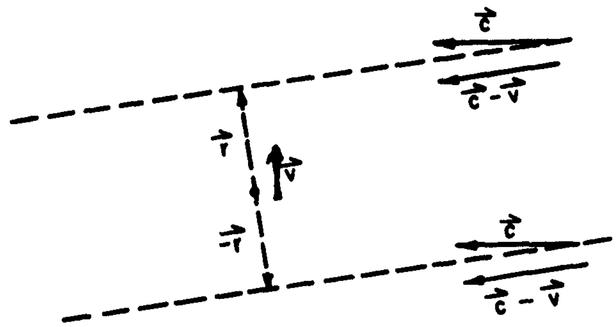

Fig. 2. $\vec{v}$: the velocity of the test body
$\vec{c}$: the velocity of two electrons
$\vec{r}$: the impact vector.

The momentum given to the test particle by the two electrons vanishes. Here the electrons are free from any of the other particles.

the density of electrons dependent only on $|c|$, we may conclude*

$$<\Delta\vec{p}>_{\Delta t} = 0 \qquad (4.1)$$

<u>Effect 2.</u> $<\Delta p\, \Delta p>_{\Delta t} = \infty$. An electron with velocity $\vec{c}-\vec{v}$ relative to the test body and impact parameter $\vec{r}$ gives the test body momentum

$$\vec{p}_i = \int_{-\infty}^{+\infty} \frac{eQ\vec{r}\, dt}{[r^2 + (\vec{c}-\vec{v})^2 t^2]^{3/2}}$$
$$= \frac{2eQ}{r|\vec{c}-\vec{v}|} \frac{\vec{r}}{r}$$

*A test body suffers a deceleration drag in average when it moves through force fields distributed at random, because the total period of deceleration is longer than the total period of acceleration. The average drag is of the order of $F^2 \ell/(Mv^2)$, where F is the average intensity of the force fields, ℓ the correlation length of the fields, M the mass of the test body, v the velocity of the body. It is assumed that $F\ell < Mv^2$. This sort of drag is considered small and is ignored.

Hence,

$$\vec{p}_i\vec{p}_i = \frac{4e^2Q^2}{r^2c^2} \tag{4.2}$$

by ignoring v^2/c^2 according to (iii). Remembering Cases 1-b and 2-b in the last section*, and considering interactions as if binary, we may calculate $<\Delta\vec{p}\Delta\vec{p}>_{\Delta t}$, by taking f for the distribution function of electrons.

$$\begin{aligned} <\Delta\vec{p}\Delta\vec{p}>_{\Delta t} &= \iiiint\limits_{r\ c} \frac{4e^2Q^2}{r^2\,c^2}\ 2\pi\, rdrfcd^3c\Delta t \\ &= 8\pi e^2Q^2\,\Delta t \iiint \frac{f}{c}\, d^3c \int \frac{dr}{r} \end{aligned} \tag{4.3}$$

Here

$$\begin{aligned} \iiint \frac{f}{c}\, d^3c &= n\left(\frac{m}{2\pi kT}\right)^{3/2} \iiint \exp\left(-\frac{m}{2kT}c^2\right) \sin\theta\, d\theta\, d\varphi \\ &= n\left(\frac{m}{2\pi kT}\right)^{3/2} 4\pi\,\frac{2kT}{m} \\ &= \frac{4n}{\sqrt{\pi}}\left(\frac{m}{2kT}\right)^{1/2} \end{aligned} \tag{4.4}$$

$$\int_{r_1}^{r_2} \frac{dr}{r} = [\log r]_{r_1}^{r_2}$$

*In the approximation of ignoring mv^2/kT, we do not need to consider $\vec{v}$ in the calculation of $<\Delta\vec{p}\,\Delta\vec{p}>_{\Delta t}$.

The result diverges* either for $r_1 = 0$ or for $r_2 = \infty$.

Approximation B. Secondly, we ignore (c), mutual interaction among electrons, and take into account (a) and (b). Here $< \Delta \vec{p} >_{\Delta t}$ and $< \Delta \vec{p} \Delta \vec{p} >_{\Delta t}$ are calculated by taking all the interactions as binary and the results are well-known:

$$
\begin{aligned}
&\lim_{r_2 \to \infty} < \Delta \vec{p} >_{\Delta t} = 0 \,[\, \log r_2 \,] = \infty, \\
&\lim \quad < \Delta \vec{p} \Delta \vec{p} >_{\Delta t} = 0 \,[\, \log r_2 \,] = \infty
\end{aligned}
\tag{4.5}
$$

In the two cases stated above, spatial fluctuations in the electron distribution are permitted with no restriction: Even the probability of all the field electrons to come together at one spot is taken into account. On the other hand, the interaction of the Vlasov type is conceived by assuming that there is

---

* Fluctuations of the electric force field in an ionized gas were calculated by the Markoff methof of random flights by J. Holtzmark [Ann. d. Physik 58, 577(1919); Physik, Zeits. 20, 162(1919), 25, 73 (1924)]. Later, Chandrasekhar and von Neumann [4] used the same method for calculating fluctuating forces exerted on a star by other stars. They considered not only the force but also its time derivative due to an assumed distribution of star velocities. In the initial formulation, they considered the correlation of the two quantities (force and its time derivative) of each field star. Because of mathematical difficulties, they abandoned the precise correlation between them and made the average of one at a given value of the other. Then they calculated the correlation period of a force by

(a given value of force)/(the average time derivative of force at the relevant value of force.)

The result apparently converges. We notice, however, that the convergence of the result is not proved by such an approximate treatment as done by Chandrasekhar et al.

no spatial fluctuation in the electron distribution. The real (feasible) condition must be between those two extreme conditions. In the following, we consider the condition. This is somehow similar to a condition of turbulence, in which dynamic characteristics of a fluid set restrictions to fluctuations appearing in the flow of the fluid.[6]

Approximation C. Finally, we conclude that the divergences (4.5) are attributed to the neglect of mutual interactions among electrons. The widely accepted solution is to consider the polarization in the distribution of electrons due to the potential field induced by each of the particles including the test body. As stated in Part I, in general, this solution does not seem plausible. Then, what is the effect of mutual interactions among field electrons by which the result converges? We note that, by the Baltzmann type binary interaction, regardless of the type of potential between two interacting particles, a limitless fluctuation in the spatial distribution of field particles is permitted to appear. In other words, each of the field particles interacting with the test particle is permitted to appear anywhere, regardless of the distribution of the other field particles. Our assertion is that such limitless fluctuations in the spatial distribution of field particles in interaction with the test particles are not permissible in an electron gas, since potential energies among field electrons prevent those electrons being accumulated at local spots. Such fluctuations may be permitted to appear only in the distribution of neighboring electrons of the test particle.

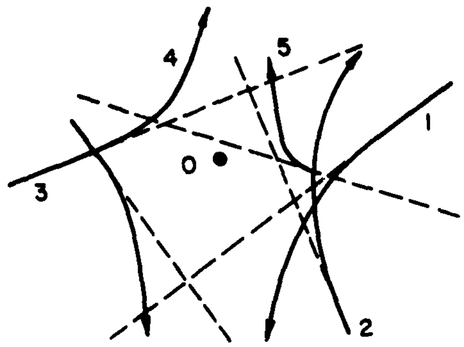

Fig. 3. 0: the test body, 1, 2, 3, ---: trajectories of electrons. If the velocity of the test body with a large mass is zero, each electron which passes by near the test body is rejected toward the outside, and anomalies occur in the distribution of electrons. Such a anomalous distribution near the test body is not Maxwellian.

Considering these, we propose the following scheme of treating electrons in an electron gas:

1. The interaction of a test electron with its mutual nearest neighbor is treated as binary interaction.
2. When observed by an observer resting on the test electron, the distribution of the mutual nearest neighbors, which come to interaction with the test electron one by one, is not uniform in general: If the test electron has no velocity relative to the average velocity of the other electrons, the density distribution of the nearest neighbors is lower with a spherical symmetry surrounding the test electron. If the test electron has a velocity relative to the average velocity of the other electrons, the density distribution to the nearest neighbors is lower behind the motion of the test electron. In other words, there is a wake behind the motion of the test electron.

3. Each of the electrons may assume a test electron. Thus each electron has its own wake.
4. Except for the wakes, all the electrons are seemingly uniformly distribute with no fluctuation in a similar manner as considered by Vlasov.
5. An electron accompanied by its mutual nearest neighbor exerts a fluctuating force on other electrons because of the anomolous distribution of its nearest neighbor.

The kinetic theoretical treatment of an electron gas according to this model will be formulated rather schematically in the following section.

SECTION V

A SCHEME OF KINETIC THEORETICAL TREATMENT OF AN ELECTRON GAS*

It is easily shown that the Liouville equation is reduced to

$$\frac{\partial F_1(x_i;t)}{\partial t} + \frac{\vec{p}_i}{m} \cdot \frac{\partial F_1(x_i;t)}{\partial \vec{q}_i}$$

$$+ \frac{1}{S\ V^s} \sum_{j----k} \cdots \sum \int \cdots \int (\vec{\mathcal{F}}_{ij} + \cdots + \vec{\mathcal{F}}_{ik}) \cdot \frac{\partial}{\partial \vec{p}_i} F_{s+1}(x_i x_j \cdots x_k;t)$$

$$dx_j \cdots dx_k = 0 \tag{5.1}$$

*This schematic presentation lacks detailed correlation functions and is not suitable for precise analyses. See Part III.

$$\left.\begin{matrix} i \\ j \\ \vdots \\ k \end{matrix}\right\} = 1, \cdots, N$$

$$i \neq j \neq \cdots \neq k.$$

We define W_{ij} as the probability of two particles, i and j, being mutual nearest neighbors: W_{ij} is a function of $(\vec{q}_i - \vec{q}_j)$ and of the number density of electrons.

By putting

$$F_2\,(ij) \equiv F_2\,(x_i x_j;t),$$

$$F_{(ij)} = W_{ij}\,F_2\,(ij), \tag{5.2}$$

$$F_{i/j} = (1-W_{ij})\,F_2\,(ij)$$

We have

$$F_2\,(ij) = F_{(ij)} + F_{i/j} \tag{5.3}$$

Since the interaction between i and j is considered weak in the domain where $F_{i/j}$ predominates, we may put $F_{i/j} = (1-W_{ij})\,[\,F_1(i)\,F_1\,(j) + F_2'(ij)\,]$ where

$$F_1(i) \equiv F_1\,(x_i;t)$$

By ignoring F_2' (ij), we have

$$F_{i/j} = (1-W_{ij})F_1(i)\,F_1(j) \tag{5.4}$$

Hence we write for (5. 3)

$$F_2(ij) = F_1(i)\, F_1(j) + W_{ij} \left[F_2(ij) - F_1(i)\, F_1(j) \right] \tag{5.5}$$

With respect to $F_3(x_i x_j x_k;t)$, we similarly define

$$\begin{aligned} &F_3\,(ijk) \equiv F_3\,(x_i x_j x_k;t), \\ &F_{(ij)/k} = W_{ij}\, F_3\,(ijk), \\ &F_{(ik)/j} = W_{ik}\, F_3\,(ijk), \\ &F_{(jk)/i} = W_{jk}\, F_3\,(ijk) \\ &F_{i/j/k} = (1 - W_{ij} - W_{ik} - W_{jk}) F_3\,(ijk) \end{aligned} \tag{5.6}$$

In the same approximation as of (5. 4), we write

$$\begin{aligned} &F_{(ij)/k} = W_{ij}\, F_2\,(ij)\, F_1\,(k), \\ &F_{(ik)/j} = W_{ik}\, F_2(ik)\, F_1\,(j), \\ &F_{(jk)/i} = W_{jk}\, F_2(jk)\, F_1(i) \\ &F_{i/j/k} = (1 - W_{ij} - W_{jk} - W_{ik}) F_1(i)\, F_1(j)\, F_1(k). \end{aligned} \tag{5.7}$$

or

$$\begin{aligned} F_3\,(ijk) = {} & F_1(i)\, F_1(j)\, F_1(k) \\ & + W_{ij} \left[F_2(ij) - F_1(i)\, F_1(j) \right] F_1\,(k) \\ & + W_{ik} \left[F_2(ik) - F_1(i)\, F_1(k) \right] F_1\,(j) \\ & + W_{jk} \left[F_2\,(jk) - F_1\,(j)\, F_1(k) \right] F_1\,(i) \end{aligned} \tag{5.8}$$

$$=F_1(i)\,F_1(j)\,F_1(k)$$
$$+F_2'(ij)\,F_1(k) + F_2'(ik)\,F_1(j) \qquad (5.9)$$
$$+F_2'(jk)\,F_1(i)$$

If we take 3 for s in Eq. (5.1) and substitute (5.9) in the equation, we obtain

$$\frac{\partial F_1(i)}{\partial t} + \frac{\vec{p}_i}{m}\cdot\frac{\partial F_1(i)}{\partial \vec{q}_i} + \Phi_1 + \Phi_2 + \Phi_3 + \Phi_4 = 0 \qquad (5.10)$$

where

$$\Phi_1 = \frac{1}{V}\sum_j \left[\int \vec{\mathcal{F}}_{ij}\,F_1(j)\,dx_j\right]\cdot\frac{\partial}{\partial \vec{p}_i}\,F_1(i) \qquad (5.11)_1$$

(the effect of Vlasov type force)

$$\Phi_2 = \frac{1}{2}\,\frac{1}{V^2}\sum_j\sum_k \iint (\vec{\mathcal{F}}_{ij} + \vec{\mathcal{F}}_{ik})\,F_2'(jk)\,dx_j\,dx_k\cdot\frac{\partial}{\partial \vec{p}_i}\,F_1(i) \qquad (5.11)_2$$

(the effect of fluctuations in the field particle distributions)

$$\Phi_3 = \frac{1}{V}\sum_j \int \vec{\mathcal{F}}_{ij}\cdot\frac{\partial}{\partial \vec{p}_i}\,F_2'(ij)\,dx_j \qquad (5.11)_3$$

(the effect of Boltzmann type interaction among mutual nearest neighbors)

$$\Phi_4 = \frac{1}{2}\,\frac{1}{V^2}\sum_j\sum_k \int\int \vec{\mathcal{F}}_{ik}\,F_1(k)\,dx_k\cdot\frac{\partial}{\partial \vec{p}_i}\,F_2'(ij)\,dx_j \qquad (5.11)_4$$

(the effect of Vlasov type force on the correlation between mutual nearest neighbors).

By taking 4, 5, --- for s in (5.1) we may have more complex effects. The detail will be investigated in Part III of this report.

SECTION VI

CONCLUDING REMARKS

1. According to the investigations carried out in Sections II and III, it is most likely that the effect of multiple interactions of a test body with field particles is the same as the effect of interactions as assumed to be binary, <u>so far as</u> the friction and diffusion of the test body in momentum space is concerned, <u>and so far as</u> the mutual interactions among field particles are ignored. The difference between two modes of interaction, multiple and binary, appears when the mutual interactions among field particles are taken into account. We induce this conclusion from the results of our various mental experiments. We might not be allowed to claim the conclusion to be a general law. At least we may be allowed to propose the conclusion as a hypothesis.

2. In view of the above conclusion, we consider an electron gas. The mutual interactions among field electrons prevent them from being in free flight. Effective fluctuations in the distribution of field electrons are: i) the fluctuations in the distribution of mutual nearest neighbors of the test electron caused by the test electron; ii) the fluctuation of each field electron caused by its mutual nearest neighbors.

3. A scheme of kinetic theoretical treatment of an electron gas based on the Lionville equation is proposed. There, the assertion (2) stated above

is formulated.

SECTION VII

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APPENDIX A

THE DERIVATION OF THE FOKKER-PLANCK EQUATION FROM THE SMOLUCHOWSKI EQUATION

By assuming $|\Delta\vec{p}| < |\vec{p}|$ and expanding functions in (2.1) in Taylor's series, we have

$$f_o(\vec{p}, t+\Delta t) = f_o(\vec{p}, t) + \frac{\partial f_o}{\partial t}\Delta t + \frac{1}{2!}\frac{\partial^2 f_o}{\partial t^2}(\Delta t)^2 + \frac{1}{3!}\frac{\partial^3 f_o}{\partial t^3} + \text{----},$$

$$f_o(\vec{p} - \Delta t,\ t) = f_o(\vec{p}, t) - \Delta\vec{p}\cdot\frac{\partial f_o}{\partial p} + \frac{1}{2!}\Delta\vec{p}\,\Delta\vec{p} : \frac{\partial^2 f_o}{\partial p\,\partial p}$$

$$- \frac{1}{3!}\Delta\vec{p}\,\Delta\vec{p} : \frac{\partial^3 f_o}{\partial\vec{p}\,\partial\vec{p}\,\partial\vec{p}} + \text{-----},$$

$$\psi(\vec{p}', t\,|\vec{p},\ t+\Delta t)\, d^3p'$$

$$= [\varphi(\vec{p},\ t;\ \Delta\vec{p},\ \Delta t) - \Delta\vec{p}\cdot\frac{\partial\varphi}{\partial\vec{p}} + \frac{1}{2!}\Delta\vec{p}\,\Delta\vec{p} : \frac{\partial^2\varphi}{\partial\vec{p}\,\partial\vec{p}}$$

$$- \frac{1}{3!}\Delta\vec{p}\,\Delta\vec{p}\,\Delta\vec{p}\ \vdots\ \frac{\partial^3\varphi}{\partial\vec{p}\,\partial\vec{p}\,\partial\vec{p}} + \text{--}\ ]\, d^3\Delta p$$

as given by (2.4). On substituting these in Eq. (2.1), we obtain

$$f_o(p, t) + \frac{\partial f_o}{\partial t}\Delta t + \frac{1}{2!}\frac{\partial^2 f_o}{\partial t^2}(\Delta t)^2 + \text{----}$$

$$= \iiint [\,(f_o - \Delta\vec{p}\cdot\frac{\partial f_o}{\partial\vec{p}} + \frac{1}{2!}\Delta\vec{p}\,\Delta\vec{p} : \frac{\partial^2 f_o}{\partial\vec{p}\,\partial\vec{p}}$$

$$- \frac{1}{3!}\Delta\vec{p}\,\Delta\vec{p}\,\Delta\vec{p}\ \vdots\ \frac{\partial^3 f_o}{\partial\vec{p}\,\partial\vec{p}\,\partial\vec{p}} + \text{---})\ \varphi(\vec{p}, t; \Delta\vec{p},\ \Delta t)$$

$$- \left(f_o - \Delta\vec{p} \cdot \frac{\partial f_o}{\partial \vec{p}} + \frac{1}{2!} \Delta\vec{p}\,\Delta\vec{p} : \frac{\partial^2 f_o}{\partial \vec{p}\,\partial \vec{p}} ----\right) \Delta\vec{p} \cdot \frac{\partial \varphi}{\partial \vec{p}}$$

$$+ \left(f_o - \Delta\vec{p} \cdot \frac{\partial f_o}{\partial \vec{p}} + \frac{1}{2!} \Delta\vec{p}\,\Delta\vec{p} : \frac{\partial^2 f_o}{\partial \vec{p}\,\partial \vec{p}} ---\right) \frac{\Delta\vec{p}\,\Delta\vec{p}}{2!} : \frac{\partial^2 \varphi}{\partial \vec{p}\,\partial \vec{p}}$$

$$---- \Big] d^3\Delta p \qquad \text{(A-1)}$$

According to (2.5) and considering that $\Delta\vec{p}$ is independent of $\vec{p}$, we have

$$\iiint \Delta\vec{p} \cdot \frac{\partial \varphi}{\partial \vec{p}}\, d^3\Delta p = \frac{\partial}{\partial \vec{p}} \cdot \iiint \Delta\vec{p}\, \varphi\, d^3\Delta p$$

$$= \frac{\partial}{\partial \vec{p}} \cdot \langle \Delta\vec{p} \rangle_{\Delta t} \qquad \text{(A-2)}$$

$$\iiint \Delta\vec{p}\,\Delta\vec{p} : \frac{\partial \varphi}{\partial \vec{p}}\, d^3\Delta p = \frac{\partial}{\partial \vec{p}} \cdot \langle \Delta\vec{p}\,\Delta\vec{p} \rangle_{\Delta t}$$

$$- - - - - - - - - - - - - - - -$$

Hence (A-1) yields

$$\frac{\partial f_o}{\partial t} \Delta t + \frac{1}{2!} \frac{\partial^2 f_o}{\partial t^2} (\Delta t)^2 ------$$

$$= - \langle \Delta\vec{p} \rangle_{\Delta t} \cdot \frac{\partial f_o}{\partial \vec{p}} + \frac{1}{2!} \langle \Delta\vec{p}\,\Delta\vec{p} \rangle_{\Delta t} : \frac{\partial^2 f}{\partial \vec{p}\,\partial \vec{p}} - \frac{1}{3!} \langle \Delta\vec{p}\,\Delta\vec{p}\,\Delta\vec{p} \rangle_{\Delta t} \vdots \frac{\partial^3 f_o}{\partial \vec{p}\,\partial \vec{p}\,\partial \vec{p}} + ---.$$

$$- f_o \frac{\partial}{\partial \vec{p}} \cdot \langle \Delta\vec{p} \rangle_{\Delta t} + \left(\frac{\partial f_o}{\partial \vec{p}}\right) \frac{\partial}{\partial \vec{p}} : \langle \Delta\vec{p}\,\Delta\vec{p} \rangle_{\Delta t}$$

$$- \frac{1}{2!} \frac{\partial^2 f_o}{\partial \vec{p}\,\partial \vec{p}} \frac{\partial}{\partial \vec{p}} \vdots \langle \Delta\vec{p}\,\Delta\vec{p}\,\Delta\vec{p} \rangle_{\Delta t} + ---$$

$$+ \frac{1}{2!} \left[+ f_o \frac{\partial^2}{\partial \vec{p}\,\partial \vec{p}} : \langle \Delta\vec{p}\,\Delta\vec{p} \rangle_{\Delta t} - \left(\frac{\partial f_o}{\partial \vec{p}}\right) \frac{\partial^2}{\partial \vec{p}\,\partial \vec{p}} \vdots \langle \Delta\vec{p}\,\Delta\vec{p}\,\Delta\vec{p} \rangle_{\Delta t} + --- \right.$$

$$- \frac{1}{3!} [\, --- \,] + ----$$

$$= - \frac{\partial}{\partial \vec{p}} \cdot (< \Delta \vec{p} >_{\Delta t} f_o) + \frac{1}{2!} \frac{\partial^2}{\partial \vec{p} \, \partial \vec{p}} : (< \Delta \vec{p} \, \Delta \vec{p} >_{\Delta t} f_o)$$

$$- \frac{1}{3!} \frac{\partial^3}{\partial \vec{p} \, \partial \vec{p} \, \partial \vec{p}} \vdots (< \Delta \vec{p} \, \Delta \vec{p} \, \Delta \vec{p} >_{\Delta t} f_o)$$

$$+ \frac{1}{4!} \frac{\partial^4}{\partial \vec{p} \, \partial \vec{p} \, \partial \vec{p} \, \partial \vec{p}} \vdots\vdots (< \Delta \vec{p} \, \Delta \vec{p} \, \Delta \vec{p} \, \Delta \vec{p} >_{\Delta t} f_o) ------ \qquad \text{(A. 3)}$$

If it is shown that

$$< \Delta \vec{p} >_{\Delta t} / \Delta t = < \Delta \vec{p} >$$

$$< \Delta \vec{p} \, \Delta \vec{p} >_{\Delta t} / \Delta t = < \Delta \vec{p} \, \Delta \vec{p} > \qquad \text{(A. 4)}$$

$$--------------$$

are independent of Δt and that

$$\frac{\partial^2 f}{\partial t^2} (\Delta t)^2 / (\frac{\partial f}{\partial t} \Delta t) < < 1,$$

$$\frac{\partial^3 f}{\partial t^3} (\Delta t)^3 / (\frac{\partial f}{\partial t} \Delta t) < < 1 \qquad \text{(A-4)}$$

$$----------$$

under certain conditions imposed by physical conditions of our interest, Eq. (A-3) is the Fokker-Planck equation.

APPENDIX B

THE FOKKER-PLANCK EQUATION

DERIVED FROM THE BOLTZMANN EQUATION

The Boltzmann equation is based on the assumption of binary collision. The collisions are Markoff processes. Hence, if most collisions are weak, both regarding the test body and the field particles, the Boltzmann equation must be reduced directly to the Fokker-Planck equation in the form as given in Appendix A. Usually,* the derivation is carried out by means of equations of moments based on the Boltzmann equation. Here is given a direct derivation: The Boltzmann equation by means of the conventional notations is written as follows:

$$\frac{df_o}{dt} = \iiint [\, f_1(\vec{p}_1{}') \, f_o(\vec{p}\,') - f_1(\vec{p}_1) \, f_o(\vec{p}) \,]$$
$$\times \; B \; db \; d\epsilon \; d^3p_1$$

Here f_1 is the distribution function of the field particles, b the impact parameter of collision between the test body and field particles, ϵ the longitudinal angle of the plane on which the trajectory of a colliding molecule is present, and B a function of b, the relative velocity and the force between the test body and the colliding molecule. There are certain relations among $\vec{p}_1{}'$, $\vec{p}\,'$, $\vec{p}_1$ and $\vec{p}$: Suppose that the collision between the test body with $\vec{p}$ and a field molecule with $\vec{p}_1$ under the condition of impact parameter b causes the test body with $\vec{p}\,'$

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*W. P. Allis: Motion of Ions and Electrons, in Handbuch der Physik, edited by S. Flügge (Springer, Berlin, 1956), Vol. 21, P. 430

and the field molecule with $\vec{p}_1'$. Then the collision between the test body with $\vec{p}'$ and the field molecule with $\vec{p}_1'$ under the same condition of impact parameter results in the test body with $\vec{p}$ and the field molecule with $\vec{p}_1$. This relation is shown by considering that the collision process, according to Newton's equations, is reversible. See Fig 4.

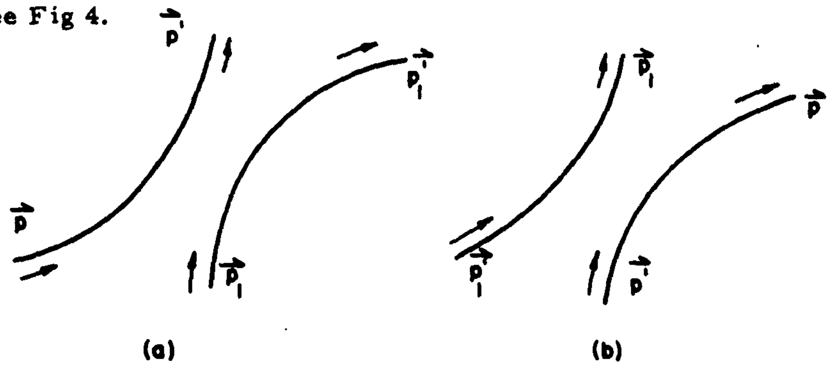

Fig. 4. In (a), a particle with momentum $\vec{p}$ and another particle with momentum $\vec{p}_1$ collide and their momenta after the collision are $\vec{p}'$ and $\vec{p}_1'$. The relation is reversible as shown in(b).

By putting

$$\Delta\vec{p} = \vec{p}_1' - \vec{p}_1 = \vec{p} - \vec{p}' \tag{B-2}$$

$\Delta\vec{p}$ is a function of $\vec{p}_1, \vec{p}, b, \theta$, and hence we may take for the independent variables

$$\Delta\vec{p},\ \vec{p},\ b,\ \theta \tag{B-3}$$

instead of

$$\vec{p}_1,\ \vec{p},\ b,\ \theta;$$

or we may write

$$\vec{p}_1 = \vec{p}_1(\Delta\vec{p},\ \vec{p},\ b,\ \theta). \tag{B-4}$$

By considering the reversible relations between $(\vec{p}_1, \vec{p})$ and $(\vec{p}_1', \vec{p}')$, we may say the following: If we define f^*_1 by

$$f^*_1 (\vec{p}; \Delta\vec{p}) = f_1 (\vec{p}_1), \tag{B-5}$$

then, considering (B-2), (B-4), we have

$$f_1 (p_1') = f_1^* (\vec{p}'; -\Delta\vec{p}) = f_1^*(\vec{p}-\Delta\vec{p}; -\Delta\vec{p}) \tag{B-6}$$

By expansion, we obtain

$$f_1(\vec{p}_1') = f_1^*(\vec{p};-\Delta\vec{p}) - \Delta\vec{p}. \frac{\partial f_1^* (\vec{p};-\Delta\vec{p})}{\partial \vec{p}}$$

$$+\frac{1}{2!} \Delta\vec{p}\,\Delta\vec{p}: \frac{\partial^2 f_1^*}{\partial\vec{p}\,\partial\vec{p}} - \frac{1}{3!} \left(\qquad \right) + \cdots \tag{B-7}$$

$$f_o(\vec{p}-\Delta\vec{p}) = f_o(\vec{p}) - \Delta\vec{p} . \frac{\partial f_o}{\partial \vec{p}} + \frac{1}{2!} \Delta\vec{p}\,\Delta\vec{p} : \frac{\partial^2 f_o}{\partial\vec{p}\,\partial\vec{p}} + \cdots. \tag{B-8}$$

Further we define B* by

$$B^* = B \, \frac{\partial (p_{1x}, p_{1y}, p_{1z})}{\partial (\Delta p_x, \Delta p_y, \Delta p_z)} \tag{B-9}$$

Equation (B-1) yields

$$\frac{d f_o}{dt} = \iiint \left[f_1^*(p - \Delta\vec{p}; \Delta\vec{p}) f_o (\vec{p} - \Delta\vec{p}) \right.$$
$$\left. - f_1^* (p; \Delta p) f_o (\vec{p}) \right] B^* \, bdbd\epsilon \, d3 \, \Delta p \tag{B-10}$$

Substituting (B-7), (B-8), and (B-9) in (B-10), we obtain

$$\frac{d f_o}{dt} = \iiint \left[f_1^*(-\Delta\vec{p}) f_o - f_1^* (\Delta\vec{p}) f_o \right]$$

$$- [f_o \Delta\vec{p} \cdot \frac{\partial f_1^*(-\Delta\vec{p})}{\partial \vec{p}} + f_1^* (-\Delta\vec{p}) \Delta\vec{p} \cdot \frac{\partial f_o}{\partial \vec{p}}]$$

$$+ \frac{1}{2!} [f_o \Delta\vec{p} \Delta\vec{p} : \frac{\partial^2 f_1^*(-\Delta\vec{p})}{\partial\vec{p}\,\partial\vec{p}} + 2 \Delta\vec{p} \Delta\vec{p} ; \frac{\partial f_o}{\partial\vec{p}} \frac{\partial f_1^*(-\Delta\vec{p})}{\partial\vec{p}}$$

$$+ f_1^*(-\Delta\vec{p}) \Delta\vec{p} \Delta\vec{p} : \frac{\partial^2 f_o}{\partial\vec{p}\,\partial\vec{p}}] - \frac{1}{3!} [\quad]$$

$$+ \cdots \Big\} B^* \, b db d\epsilon d^3 \Delta p \qquad \text{(B-11)}$$

Considering that

$$\iiint_{-\infty}^{+\infty} f_1^* (-\Delta\vec{p}) \, d^3 \Delta p = \iiint_{+\infty}^{-\infty} - f_1^* (\Delta\vec{p}) \, d^3 \Delta p$$

$$= \iiint_{-\infty}^{+\infty} f_1^* (\Delta\vec{p}) d^3 \Delta p$$

we have

$$\iiint [f_1^*(-\Delta\vec{p}) \, f_o - f_1^* (+\Delta\vec{p}) \, f_o] B^* \, b db d\epsilon d^3 \Delta p = 0$$

Further we define

$$\iiint \Delta\vec{p} \, f_1^* (-\Delta\vec{p}) \, B^* \, b db d\epsilon d^3 \Delta p = \langle \Delta\vec{p} \rangle ,$$

$$\iiint \Delta\vec{p} \, \Delta\vec{p} \, f_1^*(-\Delta\vec{p}) \, B^* \, b db d\epsilon d^3 \Delta p = \langle \Delta\vec{p} \, \Delta\vec{p} \rangle \qquad \text{(B-12)}$$

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Equation (B-11) now yields

$$\frac{df_o}{dt} = - \frac{\partial}{\partial \vec{p}} \cdot \langle \Delta p \rangle f_o + \frac{1}{2!} \frac{\partial^2}{\partial\vec{p}\,\partial\vec{p}} : \langle \Delta\vec{p} \, \Delta\vec{p} \rangle f_o - \frac{1}{3!} [\quad] + \cdots \qquad \text{(B-13)}$$

APPENDIX C

A SOLID AND ELASTIC TEST BODY IN A RAREFIED GAS CONSTITUTED OF SOLID AND ELASTIC MOLECULES

I. A Spherical Test Body (τ_r is almost zero)

We calculate $<\Delta\vec{p}>$, $<\Delta\vec{p}\,\Delta\vec{p}>$, . . . by assuming that the test body and the molecules constituting the gas are extremely hard. Due to the assumption, each collision period is extremely short and we may take Δt as short as we like. Consequently the collisions are binary.

First the test body is assumed to be a sphere of radius D/2, mass M and velocity relative to the gas $\vec{v}$, while the molecules are spheres of radius $\sigma/2$ and mass m. Further the gas is assumed to be in thermal equilibrium with temperature T. Here

$$M >> m \qquad \text{(C-1)}$$

$$v << \left(\frac{kT}{m}\right)^{\frac{1}{2}} \qquad \text{(C-2)}$$

are assumed. By taking a spherical coordinate system with origin O fixed at the center of the test body, the coordinates of the center of a molecule colliding on the surface of the test body are (R, θ, φ), where

$$R = (D + \sigma)/2$$

θ the colatitude, and φ the longitude. The number of molecules which have velocities between $\vec{c}$ and $\vec{c} + d\vec{c}$ is given by

$$fd^3c = n\left(\frac{m}{2\pi kT}\right)^{3/2} \exp\left[-\frac{m}{2kT}(c_x^2 + c_y^2 + c_z^2)\right] d^3c \qquad \text{(C-3)}$$

where $\vec{c}$ is measured by an observer at rest with respect to the laboratory (not to the test body which is moving with velocity $\vec{v}$). The number of such molecules which collide with the test body on an elementary surface area

$$dS = R^2 \sin\theta \, d\theta \, d\varphi \tag{C-4}$$

during the period between t and t + Δt is given by

$$\Delta_{cs} n = f d^3c \, |V| \cos\theta \, dS \, \Delta t \tag{C-5}$$

where

$$\vec{V} = \vec{c} - \vec{v} \tag{C-6}$$

and the z-axis (polar axis) is chosen so that the axis is parallel to $\vec{V}$ in the opposite direction.

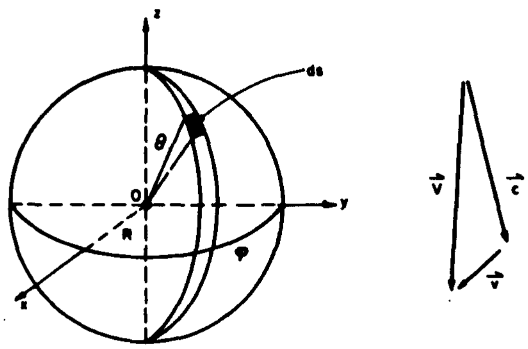


Fig. 5. $ds = R^2 \sin\theta \, d\theta d\varphi$

By carrying out the integration with respect to θ, from 0 to $\pi/2$, and φ from 0 to 2π, we obtain

$$\Delta_c n = \pi R^2 \, |V| \, f d^3c \, \Delta t \tag{C-7}$$

By choosing a rectangular coordinate system (ξ, η, ζ) so that ξ-axis is in the direction of $\vec{v}$, we have

$$\vec{v} = (v, o, o),$$

$$c_\xi - v = C_\xi ,$$

$$c_\eta = C_\eta ,$$

$$c_\zeta = C_\zeta$$

$$f = n \left(\frac{m}{2\pi kT}\right)^{3/2} \exp \left\{ - \frac{m}{2kT} \left[(C_\xi + v)^2 + C_\eta^{\,2} + C_\zeta^2 \right] \right\} \qquad \text{(C-8)}$$

$$d^3c = d^3C$$

$$V^2 = C_\xi^2 + C_\eta^2 + C_\zeta^2$$

Substituting these in (C-7), we may carry out the integration with respect to $\vec{c}$.

$$\Delta n = \pi R^2 \Delta t \iiint Cfd^3C \qquad \text{(C-9)}$$

By considering (C-2), we have

$$\exp \left[- \frac{m}{2kT} (C_\xi + v)^2 \right]$$

$$= \exp \left[- \frac{m}{2kT} C_\xi^2 \right] \times \left[1 - \frac{m}{2kT} (2C_\xi v + v^2) \right.$$

$$\left. + \frac{1}{2!} \left(\frac{m}{2kT}\right)^2 (2C_\xi v + v^2)^2 - \frac{1}{3!} (\quad) + \cdots \right] \qquad \text{(C-10)}$$

$$= \exp \left[- \frac{m}{2kT} C_\xi^2 \right] \left(1 - \frac{m}{2kT} 2 C_\xi v\right)$$

Hence

$$\Delta n = \pi R^2 \Delta t \left\{ 4 \pi n \left(\frac{m}{2 \pi kT}\right)^{3/2} \int_0^\infty C^3 \exp \left(- \frac{m}{2kT} C_\xi^2\right) dC \right.$$

$$\left. - \iiint_{-\infty}^{+\infty} \frac{mv}{kT} CC_\xi \exp \left[- \frac{m}{2kT} (C_\xi^2 + C_\eta^2 + C_\zeta^2) \right] d^3C \right\}$$

$$= 2 \pi^{1/2} n R^2 \left(\frac{2kT}{m} \right)^{1/2} \Delta t \qquad \text{(C-11)}$$

(i) $<\Delta\vec{p}>_{\Delta t}$. Each of those molecules gives the test body momentum $\delta\vec{p}$ in the direction of $\vec{R}$:

$$\delta\vec{p} = -2m|V|\cos\theta\,\vec{R}/R \tag{C-12}$$

or

$$\begin{aligned}(\delta p)_x &= \delta p\ \sin\theta\ \sin\varphi,\\ (\delta p)_y &= \delta p\ \sin\theta\ \cos\varphi,\\ (\delta p)_z &= \delta p\ \cos\theta.\end{aligned} \tag{C-13}$$

By giving the direction of $\vec{v}$ in the present coordinate system by $\theta = \theta_o$, $\varphi = 0$, we have

$$\cos\theta_o = \frac{v^2 + V^2 - c^2}{2|v|\,|V|} = \frac{v^2 - \vec{v}.\vec{c}}{|v|\,|V|} \tag{C-14}$$

and the component of $\delta\vec{p}$ in the direction of $\vec{v}$ is

$$\begin{aligned}(\delta p)_{11v} &= (\delta p)_y \sin\theta_o + (\delta p)_z \cos\theta_o\\ &= -2m\,|V|\,(\cos^2\theta\cos\theta_o + \sin\theta\cos\theta\cos\varphi\sin\theta_o)\end{aligned} \tag{C-15}$$

The momentum transferred to the test body from the field molecules with velocities between $\vec{c}$ and $\vec{c} + d\vec{c}$ in Δt has its component in the $\vec{v}$ direction

$$\begin{aligned}\Delta_c P_{11v} &= \int_{\varphi=o}^{2\pi}\int_{\theta=o}^{\pi/2} -2m\,|V|\,(\cos^2\theta\cos\theta_o + \sin\theta\cos\theta\cos\varphi\sin\theta_o)\\ &\quad .\ x\ f\ |V|\ R^2\cos\theta\sin\theta\, d\theta d\varphi\ d^3c\ \Delta t\\ &= -\pi m R^2\, V^2\cos\theta_o\ f\ d^3c\ \Delta t\\ &= -\pi m R^2\,\frac{(v^2 - \vec{v}.\vec{c})\,|\vec{c}-\vec{v}|}{|v|}\ f d^3c\ \Delta t\end{aligned} \tag{C-16}$$

For carrying out the integration with respect to $\vec{c}$, we remember (C-8) and obtain

$$\Delta P_{1v} = -\pi m R^2 \Delta t \iiint - C_\xi C \left(1 - \frac{m}{kT} C_\xi v\right)$$

$$\times n \left(\frac{m}{2\pi k T}\right)^{3/2} \exp \left\{ - \frac{m}{2kT} (C_\xi^2 + C_\eta^2 + C_\zeta^2) \right\} d^3C$$

$$= - \pi m R^2 \Delta t \; n \left(\frac{m}{2\pi kT}\right)^{3/2} \frac{m}{kT}$$

$$\times \iiint C_\xi^2 C \, v \exp \left(- \frac{m}{2kT} C^2\right) d^3C$$

By putting

$$C_\xi = C \sin \theta_1 \sin \varphi_1$$

$$d^3 C = C^2 dC \sin \theta_1 \, d\theta_1 \, d\varphi_1$$

We have

$$\iiint C_\xi^2 C \exp \left(- \frac{m}{2kT} C^2\right) d^3C$$

$$= \left[\iint \sin^3 \theta_1 \sin^2 \varphi_1 \, d\theta_1 \, d\varphi_1 \right]$$

$$\times \left[\int C^5 \exp \left(- \frac{m}{2kT} C^2\right) dC \right]$$

$$= \frac{4\pi}{3} \left(\frac{2kT}{m}\right)^3$$

Hence

$$\Delta P_{1v} = - \frac{8}{3} \pi^{1/2} m R^2 n \left(\frac{2kT}{m}\right)^{\frac{1}{2}} v \Delta t + 0 \left[v^2 \right] \tag{C-17}$$

The component perpendicular to $\vec{v}$ does obviously vanish because of the axial symmetry of the conditions with respect to the direction of $\vec{v}$. According to to the notation given in Section II, we may write

$$\langle \Delta\vec{p} \rangle_{\Delta t} = -\frac{4}{3} m \vec{v} \Delta n \tag{C-18}$$

(ii) $\langle \Delta\vec{p}\Delta\vec{p} \rangle_{\Delta t}$. We first calculate

$$\langle (\Delta p)^2 \rangle = \langle \Delta p_x \Delta p_x \rangle + \langle \Delta p_y \Delta p_y \rangle + \langle \Delta p_z \Delta p_z \rangle \tag{C-19}$$

According to (C-5) we have for this part of $\langle \Delta\vec{p}\, \Delta\vec{p} \rangle$ due to the molecules of velocities between $\vec{c}$ and $\vec{c} + d\vec{c}$

$$\langle (\Delta p)^2 \rangle_c = \Delta t\, f d^3c \int_s (\delta p)^2 |V| \cos\theta\, d S$$

where $\delta\vec{p}$ is given by (C-12). By putting

$$dS = R^2 \sin\theta\, d\theta\, d\varphi$$

We obtain

$$\langle (\Delta p)^2 \rangle_c = 2\pi m^2 R^2 V^3 f d^3C\, \Delta t$$

or by considering (C-10)

$$\langle (\Delta p)^2 \rangle_{\Delta t} = 8 \pi^{\frac{1}{2}} R^2 n m^2 \left(\frac{2kT}{m}\right)^{3/2} \Delta t + 0\, [v^2]. \tag{C-20}$$

Next, we calculate $\langle \Delta p_{11v}{}^2 \rangle$ with respect to the component Δp_{11v} in the v - direction. By considering (C-15) and (C-10), it is easily shown that

$$\langle (\Delta p_{11v})^2 \rangle = \frac{1}{3} \langle (\Delta p)^2 \rangle + 0\, [v^2] \tag{C-21}$$

Therefore, we may conclude that $\langle \Delta p_x \Delta p_x \rangle_{\Delta t}$, $\langle \Delta p_y \Delta p_y \rangle_{\Delta t}$ and $\langle \Delta p_z \Delta p_z \rangle_{\Delta t}$ are independent of $\vec{v}$, and

$$\langle \Delta p_x \Delta p_x \rangle_{\Delta t} = \langle \Delta p_y \Delta p_y \rangle_{\Delta t} = \langle \Delta p_z \Delta p_z \rangle_{\Delta t} = \frac{4}{3} m^2 \Lambda n \left(\frac{2kT}{m}\right)$$

The other components are of the order of v^2, and hence are ignored.

By means of similar ways, we may obtain $\langle\Delta\vec{p}\,\Delta\vec{p}\,\Delta\vec{p}\rangle_{\Delta t}$, $\langle\Delta\vec{p}\Delta\vec{p}\Delta\vec{p}\Delta\vec{p}\rangle_{\Delta t}$, ... An essential and common feature of those results is that they are all proportional to Δn or Δt.

II. A Cubic Test Body

In order to see in detail and more easily the mechanism causing those quantities, Δn, $\langle\Delta\vec{p}\rangle$, $\langle\Delta\vec{p}\Delta\vec{p}\rangle$, etc., we suppose the test body to be a cube with edge length (side length) D. We take the rectangular coordinate axes, x, y, z, as perpendicular respectively to three paris of faces. By assuming that the direction of $\vec{v}$ is perpendicular to one pair of faces which is, for instance, perpendicular to the x-axis

$$\vec{v} = (v, 0, 0), \tag{C-23}$$

it is easy to calculate those quantities of our present interest. The face which faces to the x-direction is denoted by S_{+x} and the face in the -x-direction by S-x. All the quantities related to S_{+x} are denoted by symbols with subscript + x, and so on. The number of molecules which collide on the side S_{+x} in Δt is given by

$$\Delta n_{+x} = \iint_{\substack{c_y \\ c_z}=-\infty}^{+\infty} \int_{c_x=-\infty}^{c_x=v} (v-c_x) f d^3c \, \Delta t D^2$$

$$= \iiint_{c_x=0}^{\infty} C_x \, n \left(\frac{m}{2\pi kT}\right)^{3/2} \exp\left\{-\frac{m}{2kT}\left[(C_x-v)^2 + C_y^2 + C_z^2\right]\right\} d^3C \Delta t D^2$$

$$= \Delta t\, D^2 n \left(\frac{m}{2\pi kT}\right)^{1/2} \left[\int_{C_x=0}^{\infty} C_x \left(1 + \frac{m}{2kT} 2 C_x v\right) \times \exp\left(-\frac{m}{2kT} C_x^2\right) d C_x \right]$$

$$= \Delta t\, D^2 n \left(\frac{m}{2\pi k T}\right)^{1/2} \left\{ \frac{kT}{m} + \frac{(\pi)^{\frac{1}{2}}}{4} v \frac{m}{kT} \left(\frac{2kT}{m}\right)^{3/2} \right\}$$

$$= \Delta t\, D^2 n \left\{ \frac{1}{2\sqrt{\pi}} \left(\frac{2kT}{m}\right)^{1/2} + \frac{v}{2} \right\} \tag{C-24}$$

where, as before, $mv^2/kT << 1$ is assumed. The number of molecules which collide on the opposite side, S-x, is given by

$$\Delta n_{-x} = \Delta t\, D^2 n \left\{ \frac{1}{2\sqrt{\pi}} \left(\frac{2kT}{m}\right)^{1/2} - \frac{v}{2} \right\} \tag{C-25}$$

We see that the number of particles colliding on S_{+x} is larger than those on S-x. On the sides perpendicular to the y- and z-axes are respectively

$$\Delta n_{+y} = \Delta n_{-y} = \Delta n_{+z} = \Delta n_{-z} \tag{C-26}$$

$$= \Delta t\, D^2 n \frac{1}{2\sqrt{\pi}} \left(\frac{2kT}{m}\right)^{1/2}$$

The total number of those molecules is

$$\Delta n = \frac{3}{\sqrt{\pi}} D^2 n \left(\frac{2kT}{m}\right)^{1/2} \Delta t \tag{C-27}$$

(i) $\langle \Delta \vec{p} \rangle_{\Delta t}$. The total momentum transferred on S + x is

$$\langle \Delta p_x \rangle_{+x} = \int\int_{c_y, c_z = -\infty}^{c_y, c_z = +\infty} \int_{C_x = -\infty}^{c_x = v} 2\, m (v - c_x)^2 f d^3 c\, \Delta t\, D^2$$

$$= \Delta t\, D^2 (-2m)\, n \left(\frac{m}{2\pi kT} \right)^{1/2} \times$$

$$\times \left[\int_{C_x = 0}^{\infty} C_x^2 \left(1 + \frac{m}{2kT}\, 2\, C_x\, v \right) \times \exp \left(- \frac{m}{2kT}\, C_x^2 \right) d\, C_x \right]$$

$$= - \frac{2mn\, D^2\, \Delta t}{\sqrt{\pi}} \left[\frac{\sqrt{\pi}}{4} \left(\frac{2kT}{m} \right) + v \left(\frac{2kT}{m} \right)^{1/2} \right] \qquad \text{(C-28)}$$

Similarly we have

$$< \Delta p_x >_{-x} = + \frac{2mnD^2 \Delta t}{\sqrt{\pi}} \left[\frac{\sqrt{\pi}}{4} \left(\frac{2kT}{m} \right) - v \left(\frac{2kT}{m} \right)^{\frac{1}{2}} \right] \qquad \text{(C-29)}$$

Hence

$$< \Delta p_x >_{\Delta t} = < \Delta p_x >_{+x} = < \Delta p_x >_{-x}$$

$$= - \frac{4mnD^2\, \Delta t}{\sqrt{\pi}} \left(\frac{2kT}{m} \right)^{1/2} v \qquad \text{(C-30)}$$

$$- < \Delta p_y >_{+y} = < \Delta p_y >_{-y} = - < \Delta p_z >_{+z} = < \Delta p_z >_{-z}$$

$$= mn\, D^2\, \Delta t \left(\frac{kT}{m} \right) \qquad \text{(C-31)}$$

and

$$< \Delta p_y >_{\Delta t} = < \Delta p_z >_{\Delta t} = 0 \qquad \text{(C-32)}$$

Summarizing these, we may write

$$< \Delta \vec{p} >_{\Delta t} = - \frac{4}{3}\, m \vec{v}\, \Delta n \qquad \text{(C-33)}$$

We see the cause of $<\Delta\vec{p}>_{\Delta t}$ in the unsymmetry appearing in Δn_{+x} and Δn_{-x} and/or in $<\Delta p_x>_{+x}$ and $<\Delta p_x>_{-x}$.

(ii) $<\Delta\vec{p}\,\Delta\vec{p}>$. Since the collisions are binary, we may write

$$< \Delta p_x \Delta p_x >_{\Delta t} = <\Delta p_x \Delta p_x >_{+x} + <\Delta p_x \Delta p_x >_{-x}$$

Here

$$<\Delta p_x \Delta p_x >_{+x} = \iiint_{c_x = \infty}^{c_x = v} 4\,m^2\,(v-c_x)^3\,fd^3c \;\; \Delta t\; D^2$$

$$= \Delta t\; D^2\; 4\,m^2 n\,\left(\frac{m}{2\pi kT}\right)^{1/2} \times \left[\int_{C_x=0}^{\infty} C_x^2\,(1+\frac{m}{2kT}\,2\,C_x\,v)\right.$$

$$\left. \times \exp\,(-\frac{m}{2kT}\,C_x^2)\,d\,C_x\right]$$

$$= 4m^2\,\Delta t\; D^2\; n\;\left(\frac{m}{2\pi kT}\right)^{1/2} \times \left[\frac{1}{2}\cdot\left(\frac{2kT}{m}\right)^2 + \frac{m}{kT}\;v\;\frac{3\sqrt{\pi}}{8}\left(\frac{2kT}{m}\right)^{3/2}\right]$$

$$= 4\,m^2\;\Delta t\;D^2\;n\;\left[\frac{1}{2\sqrt{\pi}}\left(\frac{2kT}{m}\right)^{3/2} + \frac{3}{4}\;v\;\left(\frac{m}{2kT}\right)\right]$$

Similarly

$$<\Delta p_x \Delta p_x >_{-x} = 4m^2\,\Delta t\;D^2 n\;\left[\frac{1}{2\sqrt{\pi}}\;\left(\frac{2kT}{m}\right)^{3/2} - \frac{3}{4}\;v\;\left(\frac{m}{2kT}\right)\right]$$

Hence

$$<\Delta p_x \Delta p_x > = 4m^2\Delta t\;D^2 n\,\left(\frac{2kT}{m}\right)^{3/2}$$

$$= \frac{4}{3}\;m^2\,\left(\frac{2kT}{m}\right)\;\Delta n \qquad \text{(C-34)}$$

Similarly we have

$$<\Delta p_x \Delta p_y> = <\Delta p_x \Delta p_z> = \frac{4}{3} m^2 \left(\frac{2kT}{m}\right) \Delta n \qquad \text{(C-35)}$$

Since Δp_x and Δp_y are due to two groups of particles which are independent of each other, we have

$$\begin{aligned} <\Delta p_x \Delta p_y> &= <\Delta p_x> <\Delta p_y> = 0 \\ <\Delta p_y \Delta p_z> &= <\Delta p_y> <\Delta p_z> = 0 \end{aligned} \qquad \text{(C-36)}$$

- - - - - - - - - - - -

(iii) $<\Delta\vec{p}\, \Delta\vec{p}\, \Delta\vec{p}>$. By taking advantage of easy treatments regarding this simple test body, we may have

$$<\Delta p_x \Delta p_x \Delta p_x>_{\Delta t} = <\Delta p_x \Delta p_x \Delta p_x>_{+x} \;\; <\Delta p_x \Delta p_x \Delta p_x>_{-x}$$

$$<\Delta p_x \Delta p_x \Delta p_x>_{+x} = \iiint \left[2m(v - c_x) \right] (v - c_x)\, f d^3c\, \Delta t\, D^2$$

$$= -\Delta t\, D^2\, 8m^3 n \left(\frac{m}{2\pi kT}\right)^{1/2} x$$

$$\left[\int_{C_x=0}^{\infty} C_x^4 \left(1 + \frac{m}{2kT}\, 2\, C_x v\right) \exp\left(-\frac{m}{2kT} C_x^2\right) dC_x \right]$$

$$= -\Delta t\, D^2 8m^3 n \left[\frac{3}{8}\sqrt{\pi}\left(\frac{2kT}{m}\right)^{5/2} + \left(\frac{m}{2kT}\right) 2\, v x \left(\frac{2kT}{m}\right)^3 \right] x \left(\frac{m}{2\pi kT}\right)^{1/2}$$

$$= -\Delta t\, D^2 8m^3 n \left[\frac{3}{8}\sqrt{\pi}\left(\frac{2kT}{m}\right)^{1/2} + 2v \right] \left(\frac{2kT}{m}\right)^{3/2} x \frac{1}{\sqrt{\pi}}$$

Similarly

$$<\Delta p_x \Delta p_x \Delta p_x>_{-x} = +\Delta t\, D^2 8m^3 n \left[\frac{3\sqrt{\pi}}{8} \left(\frac{2kT}{m}\right)^{1/2} - 2v \right] x \left(\frac{2kT}{m}\right)^{3/2} \frac{1}{\sqrt{\pi}}$$

Hence

$$<\Delta p_x \Delta p_x \Delta p_x>_{\Delta t} = -32\,\Delta t\, D^2 m^3 n \left(\frac{2kT}{m}\right)^{3/2} v \frac{1}{\sqrt{\pi}}$$

$$= -\frac{32}{3} m^3 \left(\frac{2kT}{m}\right) v\, \Delta n \qquad \text{(C-37)}$$

Similar calculations show that

$$<\Delta p_y \Delta p_y \Delta p_y>_{\Delta t} = <\Delta p_z \Delta p_z \Delta p_z>_{\Delta t} = 0 \qquad \text{(C-38)}$$

All the other components where Δp_i and Δp_j ($i \neq j$) appear as mixed, vanish. The reason is that each single collision is independent and each single collision does not contribute both Δp_i and Δp_j unles $i = j$.

(iv) $<\Delta\vec{p}\,\Delta\vec{p}\,\Delta\vec{p}\,\Delta\vec{p}>$. The method of calculation is similar to the preceding cases. It is simply noted that

$$<\Delta p_x \Delta p_x \Delta p_x \Delta p_x>_{\Delta t} = <\Delta p_y \Delta p_y \Delta p_y \Delta p_y>_{\Delta t}$$

$$= <\Delta p_z \Delta p_z \Delta p_z \Delta p_z>_{\Delta t} \qquad \text{(C-39)}$$

$$\propto \Delta t$$

The other components vanish.

Through those investigations, it is obvious that a molecule with $c_x = -|c_x|$, relative to the laboratory has velocity component $C_x = -|c_x| - v$ relative to the test body, and is reflected from S+x with velocity $|c_x| + v$ relative to S+x or $|c_x| + 2v$ relative to the laboratory. On the other hand, a molecule $c_x = +|c_x|$ has velocity $C_x = |c_x| - v$ relative to the test body and is

reflected from S-x with velocity $-|c_x|+v$ relative to S-x, or $-|c_x|+2v$ relative to the laboratory. We see that the test body accelerates molecules towards the direction of $\vec{v}$ and is accompanied by a wake.

We may also note that the number of molecules colliding on S+x in Δt is

$$\Delta n_{+x} = \iiint fx\,(c_x - v)\, dc_x\, dc_y\, dc_z$$

where the domains of integration are

$$c_x - v:\ -\infty \text{ to } 0,$$

$$\left.\begin{matrix} c_y \\ c_z \end{matrix}\right\}:\ -\infty \text{ to } +\infty.$$

On the other hand

$$\Delta n_{-x} = \iiint fx\,(c_x - v)\, dc_x\, dc_y\, dc_z,$$

$$c_x - v:\ 0 \text{ to } +\infty,$$

$$\left.\begin{matrix} c_y \\ c_z \end{matrix}\right\}:\ -\infty \text{ to } +\infty$$

Here all the velocity components must be given related to the laboratory.

APPENDIX D

MARKOFF'S METHOD OF RANDOM FLIGHTS

Markoff's method of random flights is summarized for the convenience of application in this article.[4]

An elementary momentum given to a test body is $\vec{p}_r$ with the probability of appearance in unit time w_r. Here we suppose that $\vec{p}_r$ and w_r are functions of a set of independent variables $q_1, q_2, \ldots, q_s$. Also we assume that there are ν of such elementary momenta.

Let us denote the probability of the total momentum being between $\vec{P}$ and $\vec{P} + d\vec{P}$ by

$$W(\vec{P})\, d^3P$$

Then

$$W(\vec{P})\, d^3P = \int \cdots \int \Lambda \prod_{r=1}^{\nu} w_r(q)\, d^s q \tag{D-1}$$

Here Δ is a function which satisfies the following conditions

$\Lambda = 1$ whenever

$$\vec{P} - \frac{1}{2}\, d\vec{P} < \sum_{r=1}^{\nu} \vec{p}_r < \vec{P} + \frac{1}{2}\, d\vec{P}, \tag{D-2}$$

$\Delta = 0$ otherwise.

It is known that Δ is given by Dirichlet's integral

$$\Delta = \frac{1}{\pi^3} \iiint \frac{\sin\left(\frac{1}{2} dP_x\, \rho_x\right)}{\rho_x}\, \frac{\sin\left(\frac{1}{2} dP_y\, \rho_y\right)}{\rho_y}\, \frac{\sin\left(\frac{1}{2} d P_z\, \rho_z\right)}{\rho_z}$$

$$\times \exp\left[i\,\vec{\rho} \cdot \left(\sum_{r=1}^{\nu} \vec{p}_r - \vec{P} \right) \right] d\rho_x\, d\rho_y\, d\rho_z \tag{D-3}$$

$$= \iiint_{\rho} \frac{d^3P}{(2\pi)^3} \exp\left[i\,\vec{\rho} \cdot \left(\sum \vec{p}_r - \vec{P} \right) \right] d^3\rho$$

Hence

$$W(\vec{P}) = \frac{1}{(2\pi)^3} \int \cdots \int \exp(-i\vec{\rho}\,.\,\vec{P})\, A(\vec{\rho})\, d^3\rho$$

$$A(\vec{\rho}) = \prod_{r=1}^{\nu} \int [\exp(i\vec{\rho}\,.\,\vec{p}_r)]\, w_r\, d^s q \qquad \text{(D-4)}$$

If p_r and w_r, as functions of the q's, do not depend on subscript r, we may have

$$A(\vec{\rho}) = \left\{ \int [\exp(i\vec{\rho}\,.\,\vec{p})]\, w\, d^s q \right\}^{\nu} \qquad \text{(D-5)}$$

For application in case of $\nu >> 1$, we may write

$$A(\vec{\rho}) = \left\{ 1 - \frac{1}{\nu} [\nu - \nu \int [\exp(i\vec{\rho}\,.\,\vec{p})]\, w\, d^s q \right\}^{\nu}$$

$$= \exp\left\{ \nu - \nu \int [\exp(i\vec{\rho}\,.\,\vec{p})]\, w\, d^s q \right\} \qquad \text{(D-6)}$$

APPENDIX E

MULTIPLE COLLISIONS OF AN ELASTIC TEST BODY IN A RAREFIED GAS

As shown in Section II of Part II, if $\nu_{\Delta t}$ is much larger than unity, we have to calculate

$$\langle \Delta p \rangle_{\Delta t} = \sum_{r=1}^{\nu_{\Delta t}} s_r\, \vec{p}_r$$

$$\langle \Delta\vec{p}\, \Delta\vec{p} \rangle_{\Delta t} = \sum \cdots \sum s_{r_1 \cdots r_\nu}\, (\vec{p}_{r_1} + \cdots + \vec{p}_{r_\nu})\, (\vec{p}_{r_1} + \cdots + \vec{p}_{r_\nu})$$

(II. 2. 26)

Since $\langle \Delta\vec{p} \rangle_{\Delta t}$ is linear with respect to $\vec{p}_{rx}$, there is no difference in $\langle \Delta\vec{p} \rangle_{\Delta t}$ between the case of binary interaction ($\nu_{\Delta t} < 1$) and the case of multiple interaction ($\nu_{\Delta t} >> 1$).

For calculating $<\Delta\vec{p}\,\Delta\vec{p}>_{\Delta t}$, in the approximation of neglecting the terms of $0\,[\frac{mv^2}{kT}]$, we may assume that the velocity of the test body relative to the gas is zero, as shown in Appendix C.

Case II-a. The test body is a spherical body submerged in a gas in thermal equilibrium.

On an elementary surface in Fig. 5,

$$dS = \sin\theta\, d\theta\, d\varphi\, R^2,$$

$$R = \frac{D}{2} + \frac{\sigma}{2},$$

we consider a local rectangular coordinate system (ξ, η, ζ) where the ζ-axis is inwardly perpendicular to dS and the ξ-and η-axes are parallel to the surface. The distribution of the field molecules is given by

$$f = n\left(\frac{m}{2\pi kT}\right)^{3/2} \exp\left[-\frac{m}{2kT}\left(c_\xi^2 + c_\eta^2 + c_\zeta^2\right)\right] \tag{E-1}$$

The number of field particles which collide on dS in Δt is given by

$$dw = f\, c_\zeta\, d^3c\, dS\, \Delta t,$$

where

$$\begin{aligned} dS &= \sin\theta\, d\theta\, d\varphi\, R^2, \\ \varphi &;\ 0 \text{ to } 2\pi, \\ \theta &;\ 0 \text{ to } \pi. \end{aligned} \tag{E-2}$$

One may obtain $\langle \nu \rangle_{\Delta t}$ directly from (E-2)

$$\langle \nu \rangle_{\Delta t} = \int d\, w$$

$$= 4\,\pi R^2\, n\, \left(\frac{kT}{2\pi m}\right) \Delta t \tag{E-3}$$

A particle gives the test body momentum $\vec{p}_r$

$$\vec{p}_r = 2\, c_\zeta\, m\vec{R}/R \tag{E-4}$$

The total momentum is

$$\vec{P} = \sum_{r=1}^{\langle \nu \rangle_{\Delta t}} \vec{p}_r \tag{E-5}$$

According to Markoff's method given in Appendix D, the probability of $\vec{P}$ being between $\vec{P}$ and $\vec{P} + d\vec{P}$ is

$$W(\vec{P})\, d^3P = \frac{1}{8\,\pi^3} \iiint_{-\infty}^{+\infty} \exp\,(-i\,\vec{\rho}\,.\,\vec{P})\, A\,(\vec{\rho}) d^3\rho\; d^3 P \tag{E-6}$$

$$A(\vec{\rho}) = \left[\int \exp\,(i\,\vec{\rho}\,.\,\vec{p})\,\frac{d\,w}{\langle \nu \rangle_{\Delta t}}\right]^{\langle \nu \rangle_{\Delta t}} \tag{E-7}$$

Since $A(\vec{\rho})$ is expected to be independent of the direction of $\vec{\rho}$ (spherical symmetry) we first calculate $A(\vec{\rho})$ by taking $\vec{\rho}$ in the direction of the z-axis.

$$A(\vec{\rho}) = \left[\iiint \exp\,(i\,\rho\,.\,2m\;c_\zeta \cos\,\theta)\;\frac{f d^3c\;\; c_\zeta\;\; R^2 \sin\,\theta\, d\,\theta\, d\,\varphi\, \Delta t}{\langle \nu \rangle_{\Delta t}}\right]^{\langle \nu \rangle_{\Delta t}}$$

$$= \left[\frac{4\,\pi R^2 n(^m/2\,\pi\, kT)^{1/2} \Delta t}{\langle \nu \rangle_{\Delta t}} \int_0^\infty \frac{\sin\,(2m\, c_\zeta\, \rho)}{2m\rho}\; \exp\,(-\,\frac{m}{2kT}\, c_\zeta{}^2)\; dc_\zeta\right]^{\langle \nu \rangle_{\Delta t}}$$

By expanding $\sin(2mc_\zeta\, p)$ in Taylor series, we have

$$\int_0^\infty \frac{\sin(2m\, c_\zeta\, \rho)}{2mp}\; \exp(-\,\frac{m}{2kT}\; c_\zeta^2)\; dc_\zeta$$

$$= \frac{kT}{m}\; \left\lfloor 1 - \frac{4}{3}\; mkT\; \rho^2 + \frac{16}{15}\, (mkT)^2\, \rho^4\right.$$

$$\left. -\,\frac{2^6}{3x5x7}\; (mkT)^3\; \rho^6\; + - - - \right\rfloor$$

Considering that $<\nu>_{\Delta t} >> 1$, we obtain for $A(\vec{\rho})$

$$A(\vec{\rho}) = \exp\left[<\nu>_{\Delta t} \left(-\frac{4}{3} mkT \rho^2 + \frac{16}{15} (mkT)^2 \rho^4 - \cdots \right) \right]$$

Remembering that $A(\vec{\rho})$ is independent of the direction of $\vec{\rho}$, we write for $A(\vec{\rho})$

$$A(\vec{\rho}) = \exp\left[- <\nu>_{\Delta t} \frac{4}{3} mkT \rho^2 \right] \times$$

$$\times \left\{ 1 + \frac{16}{15} <\nu>_{\Delta t} (mkT)^2 \rho^4 + \cdots \right\} \qquad \text{(E-8)}$$

By remembering

$$\int_0^\infty \exp(-\alpha^2 x^2) \cdot \sin \beta x \cdot x \, dx$$

$$= \frac{\sqrt{\pi}\,\beta}{4\alpha^3} \exp\left(-\frac{\beta^2}{4\alpha^2}\right)$$

$$\int_0^\infty \exp(-\alpha^2 x^2) \cdot \sin \beta x \cdot x^5 \, dx = \frac{\sqrt{\pi}\,(60\alpha^4\beta - 20\alpha^2\beta^3 + \beta^5)}{64\alpha^{11}}$$

$$\times \exp\left(-\frac{\beta^2}{4\alpha^2}\right)$$

we finally obtain

$$W(\vec{P}) = \frac{1}{8\pi^3} \iiint \exp(-i\rho P \cos\theta) A(\rho) \rho^2 d\rho \sin\theta \, d\theta \, d\varphi$$

$$= \frac{1}{2\pi^2} \int_0^\infty \frac{\sin(\rho P)}{\rho P} \rho^2 A(\rho) \, d\rho$$

$$= \left\{ \frac{1}{8\pi^{3/2} (<\nu>_{\Delta t} \frac{4}{3} mkT)^{3/2}} + 0 \left[\left(\frac{1}{<\nu>_{\Delta t}}\right)^{7/2} \right] \right\}$$

$$\times \exp\left(-\frac{3P^2}{16 <\nu>_{\Delta t} mkT}\right)$$

We may ignore the higher order terms of $(\frac{1}{\nu})$. It is shown that

$$\iiint W(\vec{P}) \, d^3P = 1 \qquad \text{(E-10)}$$

We calculate first $<\Delta\vec{p}\,\Delta\vec{p}>_{\Delta t}$:

$$<\Delta p_x\,\Delta p_x> + <\Delta p_y\,\Delta p_y> + <\Delta p_z\,\Delta p_z>$$

$$= \iiint P^2\,W(\vec{P})\,d^3P \qquad \text{(E-11)}$$

$$= 8\,mkT\,<\nu>_{\Delta t}$$

Since $W(\vec{P})$ is independent of the direction of $\vec{P}$, it is a simple matter to show that

$$<\Delta p_x\,\Delta p_x> = <\Delta p_y\,\Delta p_y> = <\Delta p_z\,\Delta p_z> = \frac{8}{3}<\nu>_{\Delta t}\,mkT \qquad \text{(E-12)}$$

$<\Delta\vec{p}\,\Delta\vec{p}\,\Delta\vec{p}>$: It is conceivable that $<\Delta\vec{p}\,\Delta\vec{p}\,\Delta\vec{p}>$ is proportional to $\vec{v}$. It is not a simple matter to calculate $<\Delta\vec{p}\,\Delta\vec{p}\,\Delta\vec{p}>$ according to the Markoff's method, since A (ρ) depends on the direction of $\vec{\rho}$. We write

$$dw = f\,(c_\zeta + v\cos\theta)\,d^3c\;ds\;\Delta t.$$

instead of (E. 2), and

$$\vec{p}_i = 2\,(C_\zeta + v\cos\theta)\,m\,\vec{R}/R$$

instead of (E. 4), by taking $\vec{v}$ in the direction of the z-axis.

$$\vec{\rho}\cdot\vec{p}_i = \rho\,|p_i|\cos\Theta$$

where Θ is a function of θ, φ and θ_ρ, φ_ρ, and the manipulation must be complicated. It is a simple matter, however, to calculate $<\Delta p_x\,\Delta p_x\,\Delta p_x\,\Delta p_x>$, by ignoring the terms of $0\,[v^2]$ by means of $W(\vec{P})$ given by (E. 6)

$$< \Delta p_x \, \Delta p_x \, \Delta p_x \, \Delta p_x > = < \Delta p_y \, \Delta p_y \, \Delta p_y \, \Delta p_y >$$

$$= < \Delta p_z \, \Delta p_z \, \Delta p_z \, \Delta p_z >$$

$$= \iiint (P \cos \theta)^4 \, W(\vec{P}) \, d^3P$$

$$= 2\pi \int_0^{\pi} \cos^4 \theta \, d\theta \int_0^{\infty} P^6 \, W(\vec{P}) \, dP$$

It is easily shown that

$$\int_0^{\infty} P^6 \, W(P) \, dP \propto \frac{< \nu >_{\Delta t}^{7/2}}{< \nu >_{\Delta t}^{3/2}} = < \nu >_{\Delta t}^{2} \qquad \text{(E-13)}$$

Cubic Test Body. If the test body is cubic, and one of the faces is perpendicular to the direction of $\vec{v}$, the calculation is very simple. Here Δn and $< \Delta \vec{p} >$ do not change from those obtained in the case of binary interactions

$$\Delta n = \frac{3}{\sqrt{\pi}} \Delta t \, D^2 n \left(\frac{2kT}{m} \right)^{1/2} ,$$

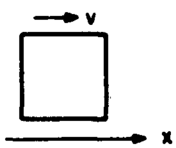
$$< \Delta \vec{p} > \doteq - \frac{4}{3} m \vec{v} \, \Delta n \qquad \text{(E-14)}$$

Case II-b_1. The particles, the number of which is given by

$$\nu = \iiint f d^3 c \quad c_\zeta \, \tau \times 6D^2 ,$$

are distributed at random on the 6 faces. Because of the steady motion of the test body perpendicular to one side, such as shown in the figure,

Fig. 6



the number of molecules which collide on S_{+x} is

$$\iint_{\left.\begin{matrix}c_y\\c_z\end{matrix}\right\}=-\infty}^{+\infty} \int_{\substack{c_x=-\infty\\v>0}}^{c_x=v} D^2 f x(-c_x+v)\, d^3c = \Delta n_{+x}$$

and the number of those which collide on the opposite side is

$$\iint_{\substack{c_y\\c_z}=-\infty}^{+\infty} \int_{c_x=v}^{c_x=+\infty} D^2 f x(c_x-v)\, d^3c = \Delta n_{-x}.$$

By ignoring the members of the order of mv^2/kt, we have

$$\Delta n_{+x} = \int_{c_y=-\infty}^{+\infty} \int_{c_x=-\infty}^{+\infty} \int_{c_x=-\infty}^{o} D^2 f x(-c_x+v)\, d^3c$$

$$= D^2 n \left(\frac{m}{2\pi kT}\right)^{\frac{1}{2}} \int_{c_x=-\infty}^{o} (-c_x+v) \exp\left(-\frac{m}{2kT} c_x^2\right) dc_x$$

$$= D^2 n \left(\frac{m}{2\pi kT}\right)^{\frac{1}{2}} \left\{ \frac{2kT}{2m} + v \frac{\sqrt{\pi}}{2} \left(\frac{2kT}{m}\right)^{\frac{1}{2}} \right\} \Delta t$$

$$= D^2 n \left\{ \frac{1}{(2\pi)^{1/2}} \left(\frac{kT}{m}\right)^{\frac{1}{2}} + \frac{v}{2} \right\} \Delta t$$

$$\Delta n_{-x} = D^2 n \left\{ \frac{1}{\sqrt{2\pi}} \left(\frac{kT}{m}\right)^{\frac{1}{2}} - \frac{v}{2} \right\} \Delta t$$

Similarly,

$$\Delta n_{+y} = \Delta n_{-y} = \Delta n_{+z} = \Delta n_{-z} = \frac{D^2 n}{(2\pi)^{\frac{1}{2}}} \left(\frac{kT}{m}\right)^{\frac{1}{2}} \Delta t$$

and

$$\Delta n = 6\, D^2 n \frac{1}{(2\pi)^{\frac{1}{2}}} \left(\frac{kT}{m}\right)^{\frac{1}{2}} \Delta t. \qquad \text{(E-15)}$$

The momentum due to Δn_{+x} in the negative direction of the x-axis is

$$\Delta p_{+x} = \iiint_{c_x = -\infty}^{c_x = v} D^2 f(-c_x + v)\, 2m\, (c_x - v)\, d^3C$$

$$= n D^2 \left(\frac{m}{2\pi kT}\right)^{\frac{1}{2}} 2m \int_{c_x = -\infty}^{0} (-c_x^2 + 2c_x v) \exp\left(-\frac{m}{2kT} c_x^2\right) dc_x$$

$$= n D^2 \left(\frac{m}{2\pi kT}\right)^{\frac{1}{2}} \times 2m \left\{ -\frac{\sqrt{\pi}}{4} \left(\frac{2kT}{m}\right)^{3/2} - 2v \frac{2kT}{2m} \right\}$$

$$= n D^2 \frac{2m}{\sqrt{\pi}} \left\{ -\frac{\sqrt{\pi}}{4} \left(\frac{2kT}{m}\right) - v \left(\frac{2kT}{m}\right)^{\frac{1}{2}} \right\}$$

Similarly

$$\Delta p_{-x} = \iiint_{c_x = 0}^{+\infty} D^2 f\, (c_x - v)^2\, 2m\, d^3c$$

$$= n D^2 \frac{2m}{\sqrt{\pi}} \left\{ + \frac{\sqrt{\pi}}{4} \left(\frac{2kT}{m}\right) - v \left(\frac{2kT}{m}\right)^{\frac{1}{2}} \right\}$$

Hence

$$\Delta \vec{p} = (\Delta p_{+x} + \Delta p_{-x}) \frac{\vec{v}}{v} = -\frac{nD^2\, 4m}{\sqrt{\pi}} \left(\frac{2kT}{m}\right)^{\frac{1}{2}} \vec{v}$$

$$= -\frac{4}{3}\, m\, \Delta n\, \vec{v} \qquad \text{(E-16)}$$

By taking the Maxwell distribution for f we have

$$\nu = \frac{3}{\sqrt{\pi}} D^2 n \left(\frac{2kT}{m}\right)^{\frac{1}{2}} \tau \qquad \text{(E-17)}$$

As before, we calculate $A(\rho)$ obtaining

$$A(\vec{\rho}) = \Bigg[\Big\{ \exp\left[i\rho_x \frac{2mc_\zeta}{\tau} \right] + \exp\left[-i\rho_x \frac{2mc_\zeta}{\tau} \right]$$

$$+ \exp\left[+i\rho_y \frac{2mc_\zeta}{\tau} \right] + \exp\left[-i\rho_y \frac{2mc_\zeta}{\tau} \right]$$

$$+ \exp\left[+i\rho_z \frac{2mc_\zeta}{\tau} \right] + \exp\left[-i\rho_z \frac{2mc_\zeta}{\tau} \right] \Big\}$$

$$\times \frac{fd^3c\; c_\zeta\; D^2 \tau}{\nu} \Bigg]^\nu$$

$$= \left\{ \left[1 - \frac{1}{2} \left(\frac{2m}{\tau}\right)^2 c_\zeta^{\;2} \frac{\rho_x^2 + \rho_y^2 + \rho_z^2}{3} \right] \frac{fd^3c\; c_\zeta 6\, D^2 \tau}{\nu} \right\}^\nu$$

$$= \left\{ \left[1 - \frac{1}{2} \left(\frac{2m}{\tau}\right)^2 \frac{\rho^2 c_\zeta^{\;2}}{3} \right] \frac{fd^3c\; c_\zeta\; 6\, D^2 \tau}{\nu} \right\}^\nu$$

$$= \exp \left\{ - \frac{4}{3} \nu\, mkT \left(\frac{\rho}{\tau}\right)^2 \right\} \qquad \text{(E-18)}$$

$$W(F) = \frac{1}{8\pi^3} \iiint \exp(-iF_x \rho_x) \exp\left[-\nu \frac{\rho_x^2}{\tau^2} \frac{4}{3} mkT \right]$$

$$\times \exp(-iF_y \rho_y) \exp\left[-\nu \frac{\rho_y^{\;2}}{\tau^2} \frac{4}{3} mkT \right]$$

$$\times \exp(-iF_z \rho_z) \exp\left[-\nu \frac{\rho_z^{\;2}}{\tau^2} \frac{4}{3} mkT \right]$$

$$\times d\rho_x\, d\rho_y\, d\rho_z$$

$$= \left(\frac{1}{2\sqrt{\pi}}\right)^3 \frac{1}{\left[\frac{\nu}{3} \left(\frac{2m}{\tau}\right)^2 \frac{kT}{m} \right]^{3/2}} \exp\left[- \frac{F^2}{\frac{4}{3} \nu \left(\frac{2m}{\tau}\right)^2 \frac{kT}{m}} \right]$$

$$\iiint F_x^{\;2}\, W(\vec{F})\, d^3F = \frac{1}{2\sqrt{\pi} \left[\frac{\nu}{3} \left(\frac{2m}{\tau}\right)^2 \left(\frac{kT}{m}\right) \right]^{\frac{1}{2}}} \int_{-\infty}^{+\infty} F_x^{\;2} \exp\left[- \frac{F_x^{\;2}}{\frac{4\nu}{3} \left(\frac{2m}{\tau}\right)^2 \frac{kT}{m}} \right] dF_x$$

$$= \frac{2\nu}{3} \left(\frac{2m}{\tau}\right)^2 \frac{kT}{m} \qquad \text{(E-19)}$$

Hence

$$<\Delta p_x \, \Delta p_x> = \frac{8}{3} \, \nu \, mkT \, \Delta t/\tau = \frac{8}{3} \, mkT \, \Delta n \qquad \text{(E-20)}$$

Here again we see that

$$<\Delta p_x \, \Delta p_x \, \Delta p_x \, \Delta p_x> \propto \nu^2 \qquad \text{(E-21)}$$

Case II-b_2. On each side, the number of particles is assigned as $\frac{\nu}{6}$ due to some physical condition. In this case, we have to calculate $<\Delta \vec{p} \, \Delta \vec{p}>$ on each side. For example, by taking the side which is perpendicular to the x-axis on the positive side, we have only one finite component, $<\Delta p_x \, \Delta p_x>_+$. For $A(\rho)$ we have

$$A(\rho_x)_+ = \exp \left\{ - \frac{\nu}{6} \left(\frac{2m}{\tau}\right)^2 \frac{kT}{m} \rho_x^{\,2} - i \, \frac{\nu}{6} \sqrt{\pi} \, \frac{m}{\tau} \left(\frac{2kT}{m}\right)^{\frac{1}{2}} \rho_x \right\} \qquad \text{(E-22)}$$

$$W(F_x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \exp\left(-i \, F_x \, \rho_x\right) \, A(\rho_x) \, d\rho_x$$

$$= \frac{1}{2\sqrt{\pi}} \cdot \frac{1}{\left[\frac{\nu}{6} \left(\frac{2m}{\tau}\right)^2 \frac{kT}{m} \right]^{\frac{1}{2}}} \exp \left\{ - \frac{\left[F_x + \nu\sqrt{\pi} \, \frac{m}{\tau} \left(\frac{2kT}{m}\right)^{\frac{1}{2}} \right]^2}{\frac{4\nu}{6} \left(\frac{2m}{\tau}\right)^2 \frac{kT}{m}} \right\} \qquad \text{(E-23)}$$

Noting that

$$\frac{\left[\nu\sqrt{\pi} \, \frac{m}{\tau} \left(\frac{2kT}{m}\right)^{\frac{1}{2}} \right]^2}{4\nu \left(\frac{2m}{\tau}\right)^2 \frac{k\,T}{m}} = \frac{\pi}{8} \nu >> 1,$$

$F_x^{\,1} = F_x + \nu\sqrt{\pi} \cdot \frac{m}{\tau} \left(\frac{2kT}{m}\right)^{\frac{1}{2}}$ ranges from $-\infty$ to $+\infty$ as F_x changes from $-\infty$ to 0.

Hence we have

$$\int_0^\infty F_{+x}\, W(F_x)_+\, dF_x = -\sqrt{\pi}\ \frac{\nu}{6}\ \frac{m}{\tau}\ \left(\frac{2kT}{m}\right)^{\frac{1}{2}}$$

$$\int_0^\infty F_{+x}^2\ W(F_x)_+\ dF_x = \frac{\nu}{3}\left(\frac{2m}{\tau}\right)^2 \frac{kT}{m} + 2\pi\left(\frac{\nu}{6}\right)^2\left(\frac{m}{\tau}\right)^2 \frac{kT}{m} \qquad \text{(E 24)}$$

and for the opposite side

$$\int_0^\infty F_{-x}\, W(F_x)_-\ dF_x = \sqrt{\pi}\ \frac{\nu}{6}\left(\frac{m}{\tau}\right)\left(\frac{2kT}{m}\right)^{\frac{1}{2}}$$

$$\int_0^\infty F_{-x}^2\ W(F_x)_-\ dF_x = \frac{\nu}{3}\left(\frac{2m}{\tau}^2\right) \frac{2kT}{m} + 2\pi\left(\frac{\nu}{6}\right)^2\left(\frac{m}{\tau}\right)^2 \frac{kT}{m} \qquad \text{(E 25)}$$

For calculating the total in the x-direction, we put

$$<\Delta p_x\, \Delta p_x>_{\Delta t} = \tau^2 < (F_{+x} + F_{-x})^2 > x\ \frac{\Delta t}{\tau}$$

$$= \tau\Delta t\ (<F_{+x}^2> + <F_{-x}^2> + 2<F_{+x}F_{-x}>)$$

Since F_{+x} and F_{-x} are independent events, we nave

$$<\Delta p_x\, \Delta p_x>_{\Delta t} = \tau\Delta t\ (<F_{+x}^2> + <F_{-x}^2> + 2<F_{+x}><F_{-x}>)$$

$$= \frac{2}{3}\ \nu\left(\frac{2m}{\tau}\right)^2 \frac{kT}{m}\frac{\Delta t}{\tau}$$

$$= \frac{8}{3}\ mkT\ \Delta n \qquad \text{(E 26)}$$

Case II-b_3. We now assume that the distribution of field particles in the configuration space is at random. It is assumed, however, that each particle has the same momentum components

$$|p_{ix}| = |p_{iy}| = |p_{iz}| = \frac{\iiint m\,|c_x|\,fd^3c}{\iiint fd^3c}$$

$$= m\left(\frac{m}{2\pi kT}\right)^{\frac{1}{2}}\left(\frac{kT}{m}\right)$$

$$= \frac{1}{(2\pi)^{\frac{1}{2}}}\, m\left(\frac{kT}{m}\right)^{\frac{1}{2}}$$

$$= \left(\frac{mkT}{2\pi}\right)^{\frac{1}{2}} \tag{E. 27}$$

In this case we obtain

$$\langle \nu \rangle_{\Delta t} = 6nD^2\,\Delta t\left(\frac{mkT}{2\pi}\right)^{\frac{1}{2}} \Big/ m$$

$$= \frac{6D^2 n}{(2\pi)^{\frac{1}{2}}}\left(\frac{kT}{m}\right)^{\frac{1}{2}}\Delta t \tag{E 28}$$

$$\langle \Delta p_x \rangle_{\Delta t} = D^2 n\left\{\frac{1}{(2\pi)^{\frac{1}{2}}}\left(\frac{kT}{m}\right)^{\frac{1}{2}} + v\right\}\Delta t$$

$$\times\left\{\left(\frac{mkT}{2\pi}\right)^{\frac{1}{2}} + m\,v\right\}\times 2$$

$$- D^2 n\left\{\frac{1}{(2\pi)^{\frac{1}{2}}}\left(\frac{kT}{m}\right)^{\frac{1}{2}} - v\right\}\Delta t\,\cdot$$

$$\times\left\{\left(\frac{mkT}{2\pi}\right)^{\frac{1}{2}} - m\,v\right\}\times 2$$

$$= -4\left\{\frac{1}{\sqrt{2\pi}}\left(\frac{kT}{m}\right)^{\frac{1}{2}} m + \left(\frac{mkT}{2\pi}\right)^{\frac{1}{2}}\right\} v\Delta t\, D^2 n$$

$$= -8\left(\frac{mkT}{2\pi}\right)^{\frac{1}{2}} v\Delta t\, D^2 n$$

$$= -\frac{4}{3}\, m\,\langle \nu \rangle_{\Delta t}\, v \tag{E 29}$$

In this case

$$A(\vec{\rho}) = \left\{ 2\cos\left[2\rho_x \left(\frac{mkT}{2\pi}\right)^{\frac{1}{2}}\right] + 2\cos\left[2\rho_y\left(\frac{mkT}{2\pi}\right)^{\frac{1}{2}}\right]\right.$$

$$\left. + 2\cos\left[2\rho_z\left(\frac{mkT}{2\pi}\right)^{\frac{1}{2}}\right]\right\}^{\nu} \left(\frac{\nu/6}{\nu}\right)^{\nu}$$

$$= \exp\left\{ -\left(\frac{4mkT}{2\pi}\right) \frac{1}{2}\left(\frac{\rho^2}{3}\right)\nu\right\}$$

$$= \exp\left\{ -\nu\rho^2 \frac{mkT}{3\pi}\right\} \tag{E-30}$$

$$W(P_x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \exp(-i P_x \rho_x) \exp\left[-\nu\rho_x^2 \frac{mkT}{3\pi}\right] d\rho_x$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \cos(P_x\rho_x)\exp\left[-\nu\frac{mkT}{3\pi}\rho_x^2\right] d\rho_x$$

$$= \frac{1}{2\pi}\frac{\sqrt{\pi}}{\left(\nu\frac{mkT}{3\pi}\right)^{\frac{1}{2}}} \cdot \exp\left[\frac{-P_x^2}{4\nu\frac{mkT}{3\pi}}\right]$$

$$= \frac{1}{2\pi^{\frac{1}{2}}}\frac{1}{\left(\nu\frac{mkT}{3\pi}\right)^{\frac{1}{2}}} \exp\left[\frac{-P_x^2}{4\nu\frac{mkT}{3\pi}}\right] \tag{E-31}$$

$$<\Delta p_x \Delta p_x> = \frac{1}{2\pi^{\frac{1}{2}}}\frac{1}{\left(\nu\frac{mkT}{3\pi}\right)^{\frac{1}{2}}}\frac{\sqrt{\pi}}{2}\left(4\nu\frac{mkT}{3\pi}\right)^{3/2}$$

$$= \frac{2}{3\pi}\nu\, mkT$$

$$\nu = <\nu>_{\Delta t} \tag{E-32}$$

Case II-b_4. Each particle exerts the average force and $\frac{\nu}{6}$ particles are assigned on each side. It is obvious that Δp_x has no fluctuation and $<\Delta\vec{p}\,\Delta\vec{p}>$ vanishes.

APPENDIX F

USEFUL INTEGRALS

The integrals which often appear in the present manipulation are listed as follows [7, 8]:

$$\int_0^\infty X^{2n} \exp(-\lambda X^2)\, dX = \frac{1, 3, ---, (2n-1)}{2^{n+1}} \left(\frac{\pi}{\lambda^{2n+1}}\right)^{\frac{1}{2}},$$

$$\int_0^\infty X^{2n+1} \exp(-\lambda X^2)\, dX = \frac{n!}{2\lambda^{n+1}},$$

$$\int_0^\infty \exp(-\lambda X^2)\, dX = \frac{1}{2}\left(\frac{\pi}{\lambda}\right)^{\frac{1}{2}},$$

$$\int_0^\infty X \exp(-\lambda X^2)\, dX = \frac{1}{2\lambda},$$

$$\int_0^\infty X^2 \exp(-\lambda X^2)\, dX = \frac{1}{4}\left(\frac{\pi}{\lambda^3}\right)^{\frac{1}{2}},$$

$$\int_0^\infty X^3 \exp(-\lambda X^2)\, dX = \frac{1}{2\lambda^2},$$

$$\int_0^\infty X^4 \exp(-\lambda X^2)\, dX = \frac{3}{8}\left(\frac{\pi}{\lambda^5}\right)^{\frac{1}{2}}$$

$$\int_0^\infty X^5 \exp(-\lambda X^2)\, dX = \frac{1}{\lambda^3}$$

$$\int_0^\infty X^6 \exp(-\lambda X^2)\, dX = \frac{15}{16}\left(\frac{\pi}{\lambda^7}\right)^{\frac{1}{2}}$$

$$\int_0^\infty X^7 \exp(-\lambda X^2)\, dX = \frac{3}{\lambda^4}.$$

$$\int_0^\infty \exp(-p^2 X^2) \cos(qX)\, dX = \frac{1}{2} \frac{\pi^{\frac{1}{2}}}{p} \exp\left[-q^2/4p^2\right]$$

$$\int_0^\infty \exp(-p^2 X^2) \sin(qX). X dX = \frac{\pi^{\frac{1}{2}} q}{4p^3} \exp\left[-q^2/4p^2\right]$$

$$\int_0^\infty \exp(-p^2 X^2) \cos(qX).\ X^2\, dX = \frac{(2p^2 - q^2)\ \pi^{\frac{1}{2}}}{8p^5} \exp\left[-q^2/4p^2\right]$$

$$\int_0^\infty \exp(-p^2 X^2) \sin(qX).\ X^3\, dX = \frac{(6p^2 q - q^3)\ \pi^{\frac{1}{2}}}{16p^7} \exp\left[-q^2/4p^2\right]$$

$$\int_0^\infty \exp(-p^2 X^2) \cos(qX).\ X^4 dX = \frac{(12p^4 - 12p^2q^2 + q^4)\ \pi^{\frac{1}{2}}}{32p^9} \exp\left[-q^2/4p^2\right]$$

$$\int_0^\infty \exp(-p^2 X^2) \sin(qX). X^5 dX = \frac{(60p^4 q - 20p^2 q^3 + q^5)\ \pi^{\frac{1}{2}}}{64p^{11}} \exp\left[-q^2/4p^2\right]$$

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Unclassified
Security Classification

DOCUMENT CONTROL DATA - R&D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY (Corporate author) Polytechnic Institute of Brooklyn Graduate Center Route 110, Farmingdale, New York 11735	2a. REPORT SECURITY CLASSIFICATION Unclassified 2b. GROUP

3. REPORT TITLE
A KINETIC THEORETICAL INVESTIGATION OF A FULLY IONIZED GAS
Part II - Some Aspects of Multiple Collisions

4. DESCRIPTIVE NOTES (Type of report and inclusive dates)
Interim Report

5. AUTHOR(S) (Last name, first name, initial)
Toyoki Koga

6. REPORT DATE January 1965	7a. TOTAL NO. OF PAGES 72	7b. NO. OF REFS 8

8a. CONTRACT OR GRANT NO. Nonr 839(38)	9a. ORIGINATOR'S REPORT NUMBER(S) PIBAL Report No. 863
b. PROJECT NO. NR 061-135	
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13. ABSTRACT

The Brownian motion of a test body due to multiple interactions with field particles is investigated within or almost within the framework of Markoff's processes. First, Markoff's processes are studied as presenting such multiple interactions. Based on the study and by means of Markoff's method of random flights, we investigate the Brownian motion of an elastic test body submerged in a rarefied gas constituted of elastic molecules, under the condition that mutual interactions among field particles are negligible. It is shown that there is no difference in effect between temporal repetitions of random binary collisions and multiple collisions (random binary collisions superposed at one moment of time), so far as the friction and diffusion of the test body in momentum space are concerned. The situation is similar when a test body with electric charge is submerged in an electron gas, if the mutual interactions among electrons are ignored. It is not feasible, however, to ignore those mutual interactions of field electrons and to represent electronic multiple interactions by temporal repetitions of random binary interactions, each of which takes place independently: Fluctuations of limitlessly large amplitudes in the spatial distribution of electrons, which may possibly take place in this approximation, do not seem realistic, because a limitless concentration of potential energy accompanying a concentration of electrons in a local spot cannot be permitted. Amplitudes of such fluctuations and/or microscopic disturbances must have a certain maximum limit. (The situation does not change even when the interaction force law is of the Debye-Huckel type.) A kinetic theoretical scheme of treating fully ionized gas in light of this fact is proposed.

DD FORM 1 JAN 64 1473

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